Quiz for August 31, 2005

Let \( f(x) = 3x^2 + 5x - 2 \) for \( 0 \leq x \).

a. Find \( f^{-1}(x) \).

b. Find the domain of \( f^{-1}(x) \).

c. Verify that \( f(f^{-1}(x)) = x \) for all \( x \) in the domain of \( f^{-1}(x) \).

d. Verify that \( f^{-1}(f(x)) = x \) for all \( x \) in the domain of \( f(x) \).

\textbf{ANSWER:} Let \( y = f^{-1}(x) \). We know that \( f(y) = x \) and that \( y \) is in the domain of \( f \). In other words, \( 3y^2 + 5y - 2 = x \) and \( 0 \leq y \). Re-write the equation to get \( 3y^2 + 5y - 2 - x = 0 \). This is a quadratic equation of the form \( ay^2 + by + c = 0 \), with \( a = 3 \), \( b = 5 \), and \( c = -2 - x \). Apply the quadratic formula. The solution of \( ay^2 + by + c = 0 \) is \( y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). So for us,

\[
y = \frac{-5 \pm \sqrt{25 - 12(-2 - x)}}{6} = \frac{-5 \pm \sqrt{49 + 12x}}{6}
\]

Our \( y \) MUST be at least zero; so we need, to take + rather than - . Also, we must have

\[
0 \leq \frac{-5 + \sqrt{49 + 12x}}{6}
\]

\[
0 \leq -5 + \sqrt{49 + 12x}
\]

\[
5 \leq \sqrt{49 + 12x}
\]

\[
25 \leq 49 + 12x
\]

\[
-24 \leq 12x
\]

\[
-2 \leq x.
\]

Our answer to (a) is \( f^{-1}(x) = \frac{-5 + \sqrt{49 + 12x}}{6} \). Our answer to (b) is \( -2 \leq x \).

(c) Take \(-2 \leq x \). Observe that

\[
f(f^{-1}(x)) = f \left( \frac{-5 + \sqrt{49 + 12x}}{6} \right)
\]

\[
= 3 \left( \frac{-5 + \sqrt{49 + 12x}}{6} \right)^2 + 5 \left( \frac{-5 + \sqrt{49 + 12x}}{6} \right) - 2
\]
\[= \frac{(-5 + \sqrt{49 + 12x})^2}{12} + 5 \left(\frac{-5 + \sqrt{49 + 12x}}{6}\right) - 2\]
\[= \frac{1}{12} \left[(-5 + \sqrt{49 + 12x})^2 + 10(-5 + \sqrt{49 + 12x}) - 24\right]\]
\[= \frac{1}{12} \left[25 - 10\sqrt{49 + 12x} + 49 + 12x + 10(-5 + \sqrt{49 + 12x}) - 24\right]\]
\[= \frac{1}{12} [25 + 49 + 12x - 50 - 24] = x. \checkmark\]

(d) Take \(0 \leq x\). Observe that

\[f^{-1}(f(x)) = f^{-1}(3x^2 + 5x - 2) = \frac{-5 + \sqrt{49 + 12(3x^2 + 5x - 2)}}{6}\]
\[= \frac{-5 + \sqrt{49 + 36x^2 + 60x - 24}}{6} = \frac{-5 + \sqrt{36x^2 + 60x + 25}}{6} = \frac{-5 + \sqrt{(6x + 5)^2}}{6}\]
\[= \frac{-5 + |6x + 5|}{6}\]

We know that \(0 \leq x\); so, \(0 \leq 6x + 5\); so the most recent expression is

\[= \frac{-5 + 6x + 5}{6} = \frac{6x}{6} = x. \checkmark\]