PRINT Your Name: ___________________________ Section: ___________
There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. [CIRCLE] your answer. NO CALCULATORS! You might find the following formulas to be useful:

\[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \]

1. State the Mean Value Theorem. If \( f \) is a differentiable function for \( a \leq x \leq b \), then there exists a number \( c \) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a}. \]

2. Define the definite integral. For each partition of the interval \( a \leq x \leq b \) as \( x_0 = x_0 \leq x_1 \leq ... \leq x_n = b \), let \( M_i \) be the maximum of \( f \) on \([x_{i-1}, x_i]\) and \( m_i \) the minimum of \( f \) on \([x_{i-1}, x_i]\) for \( i = 1, 2, ..., n \). Let \( U \) be the upper sum \( U = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \) and \( L \) be the lower sum \( L = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \). If there is exactly one number between each upper sum and every lower sum, then the definite integral \( \int_{a}^{b} f(x) \, dx \) exists.

3. Find \( \int x (2x^2 + \frac{1}{x}) \, dx \). (Check your answer.)

\[ \int x (2x^2 + \frac{1}{x}) \, dx = \int (2x^3 + 1) \, dx = \frac{x^4}{2} + x + C \]
4. Find \( \int (\cos^4 x)(x^2 \sin x^3) \, dx \). (Check your answer.)

Let \( u = \cos x^3 \)
\[
du = -3x^2 \sin x^3 \, dx
\]
\[
= \int u^4 (-\frac{1}{3}) \, du
\]
\[
= \frac{u^5}{15} + C
\]
\[
= -\frac{1}{15} \cos^5 x^3 + C
\]

5. Find \( \int x \sqrt{x + 1} \, dx \). (Check your answer.)

Let \( u = x + 1 \)
\[
du = dx
\]
\[
u - 1 = x
\]
\[
\int u^\frac{3}{2} - u^\frac{1}{2} \, du
\]
\[
= \frac{2}{3} u^\frac{5}{2} - \frac{2}{3} u^\frac{3}{2} + C
\]
\[
= \frac{2}{3} (x+1)^\frac{5}{2} - \frac{2}{3} (x+1)^\frac{3}{2} + C
\]

6. Solve the Initial Value Problem \( \frac{dy}{dt} = t^2 y^2 \), \( y(2) = 1 \). (Check your answer.)

\[
\int y^{-2} \, dy = \int t^2 \, dt
\]
\[
-\frac{1}{y} = \frac{t^3}{3} + C
\]
\[
-\frac{1}{y} = \frac{t^3}{3} + C
\]
\[
y = \frac{-1}{\frac{t^3}{3} + C}
\]
\[
y = \frac{-1}{\frac{t^3}{3} - \frac{5}{3}}
\]
\[
y = -\frac{4}{t^3 - 5}
\]

\[
y = \frac{-4}{t^3 - \frac{5}{3}}
\]
7. Consider the region $A$, which is bounded by the $x -$ axis, $y = \frac{2}{3}x + 1$, $x = 1$, and $x = 2$. Consider 50 rectangles, all with base $1/50$, which UNDER estimate the area of $A$. How much area is inside the 50 rectangles? (You must answer the question I asked. I expect an exact answer.)

$$\begin{align*}
\text{Area} &= \frac{1}{50} \left( (1 + \frac{1}{50} - 1)^2 + (1 + \frac{2}{50} - 1)^2 + \cdots + (1 + \frac{49}{50} - 1)^2 \right) \\
&= \frac{1}{50} \left( \frac{0^2}{50^2} + \frac{1^2}{50^2} + \cdots + \frac{49^2}{50^2} \right) \\
&= \frac{1}{50^3} \left( 1^2 + 2^2 + \cdots + 49^2 \right) \\
&= \frac{1}{50^3} \left( \frac{(49)(50)(99)}{6} \right) \\
&= \frac{(49)(99)}{6(50)^2}
\end{align*}$$
8. Let $f(x) = x^{5/3} - x^{2/3}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

\[ f'(x) = \frac{5}{3} x^{2/3} - \frac{2}{3} x^{-1/3} = \frac{1}{3} x^{-1/3} (5x - 2) \]
\[ f''(x) = \frac{10}{9} x^{-1/3} + \frac{2}{9} x^{-4/3} = \frac{2}{9} x^{-4/3} (5x + 1) \]

\[ c < f' < 0 \quad f'' \text{pos} \quad f'' \text{pos} \quad f'' \text{pos} \]

inc for $x < 0$ also for $\frac{2}{5} < x$
dec for $0 < x < \frac{2}{5}$
c.u. for $-\frac{1}{5} < x$
c.d. for $x < -\frac{1}{5}$
local max at $0$
local min at $(\frac{2}{5}, f(\frac{2}{5}))$
local min at $(-\frac{1}{5}, f(-\frac{1}{5}))$
No v. or h. asymptotes.
9. Find the points on the curve \( y = 10 - x^2 \) which are closest to the point \((0, 0)\).

Let \((x, y)\) be a point on the parabola. We need to minimize \( f = x^2 + y^2 \)
so \( f = 10 - y + y^2 \) with \( y \leq 10 \)

\[
f' = -1 + 2y
\]
\[
f' = 0 \text{ when } y = \frac{1}{2} \quad (\text{This obviously is the min of } f(y))
\]

because \( f(y) \) is a parabola which opens downwards.

So the closest points are \( (\pm \sqrt{9}, \frac{1}{2}) \).

10. A 30-foot ladder is leaning against a wall. If the bottom of the ladder is pulled along the level pavement directly away from the wall at 3 feet per second, how fast is the top of the ladder moving down the wall when the foot of the ladder is 5 feet from the wall?

\[
\frac{d}{dt} (x^2 + y^2) = (30)^2
\]

\[
\frac{dy}{dt} \bigg|_{x=5} = \frac{-x}{y} \frac{dx}{dt}
\]

\[
\frac{dy}{dt} \bigg|_{x=5} = \frac{-5}{\sqrt{900-25}} = \frac{6}{5}
\]