PRINT Your Name:  

There are 13 problems on 7 pages. Problems 1 and 2 are each worth 6 points. Each of the other problems is worth 8 points. SHOW your work.  CIRCLE your answer. You might find the following formulas to be useful:

\[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \]

NO CALCULATORS!

1. State both parts of the Fundamental Theorem of Calculus.

   Let \( f(x) \) be continuous for \( a \leq x \leq b \)
   
   a) If \( A(x) = \int_{a}^{x} f(t) \, dt \), then \( A'(x) = f(x) \).
   
   b) If \( F(x) \) is any antiderivative of \( f(x) \), then
   
   \[ \int_{a}^{b} f(t) \, dt = F(b) - F(a) \]

2. Define the definite integral.

   Let \( f(x) \) be a function which is defined for \( a \leq x \leq b \). For each partition \( P: a = x_0 \leq x_1 \leq \ldots \leq x_n = b \), let
   
   \[ U_P = M_1 (x_1 - x_0) + M_2 (x_2 - x_1) + \cdots + M_n (x_n - x_{n-1}) \]
   
   and
   
   \[ L_P = m_1 (x_1 - x_0) + m_2 (x_2 - x_1) + \cdots + m_n (x_n - x_{n-1}) \]

   where \( M_i \) is the maximum of \( f(x) \) on \( [x_{i-1}, x_i] \) and
   
   \( m_i \) is the minimum of \( f(x) \) on \( [x_{i-1}, x_i] \).

   If there is exactly one number \( \# \) with
   
   \[ L_P \leq \# \leq U_P \]

   for all partitions \( P \),

   then \( \# \) is called the definite integral of \( f \) on \( [a,b] \) and it is denoted \( \int_{a}^{b} f(x) \, dx \).
3. Let \( y = \sqrt{x \cos^3(4x^2 + 3) + \sin^4(x)} \). Find \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = \frac{-3x \cos^2(4x^2 + 3) \sin(4x^2 + 3) 8x + \cos^3(4x^2 + 3) + 4 \sin^3 x \cos x}{2 \sqrt{x \cos^3(4x^2 + 3) + \sin^4 x}}
\]

4. Find \( \int \frac{2}{x^2} + \sin(2x) \, dx \).

\[
= -\frac{2}{x} - \frac{\cos 2x}{2} + C
\]

5. Find \( \int \frac{\sin x \cos x}{\sqrt{2 \sin^2 x + 1}} \, dx \).

\[
\int \frac{1}{4} u^{-\frac{1}{2}} \, du = \frac{1}{4} \frac{1}{2} u^{\frac{1}{2}} + C
\]

\( u = 2 \sin^2 x + 1 \)

\( du = 4 \sin x \cos x \, dx \)

\[
= \frac{1}{8} (2 \sin^2 x + 1)^{\frac{1}{4}} + C
\]
6. Let \( f(x) = \frac{x^2 - 2x + 4}{x - 2} \). Where is \( f(x) \) increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of \( y = f(x) \). Find all vertical and horizontal asymptotes. Graph \( y = f(x) \).

\[
\frac{f'}{f'} = \frac{(x-2)(2x-1) - (x^2-2x+4)}{(x-2)^2} = \frac{2x^2 - 6x + 4 - (x^2 - 2x + 4)}{(x-2)^2}
\]

\[
= \frac{x^2 - 4x}{(x-2)^2} = \frac{x}{(x-2)^2}
\]

\[
\frac{f''}{f''} = \frac{(x-2)^2(2x-4) - (x^2-4x)^2}{(x-2)^4} = \frac{(x-2)(2x-4)(x-4) - 2(x^2-4x)}{(x-2)^4}
\]

\[
= \frac{2x^2 - 8x + 8 - 2x^2 + 8x}{(x-2)^3}
\]

\[
= \frac{8}{(x-2)^3}
\]

\[
\begin{align*}
\lim_{x \to 2^+} f' &= +\infty \\
\lim_{x \to 2^-} f' &= -\infty \\
\lim_{x \to \infty} f' &= +\infty \\
\lim_{x \to -\infty} f' &= -\infty
\end{align*}
\]

\[
f(4) = 6, \quad f(0) = -2
\]

\[
V. \text{asy} \quad (0, 4) \\
N. \text{asym} \quad \text{h. asym} \\
\text{loc max} \quad (4, 6) \\
\text{loc min} \quad (0, -2) \\
\text{No st. int.} \\
\text{Int: } 0 < x < 2, 0 \text{ also for } x < 0 \\
\text{dec: } 0 < x < 2, 0 \text{ also for } 2 < x < 4 \\
\text{cu dec } x < 2 \\
\text{cd inc } x > 2
\]
7. The surface area of a cube is growing at the constant rate of 1000 square inches per second. How fast is the volume growing when each edge is 5 inches long?

Let \( l \) = length of each edge \( S = 6l^2 \); \( \frac{dS}{dt} = 1000 \)

Find \( \frac{dV}{dt} \) when \( l = 5 \)

\[
\frac{dS}{dt} = 12l \frac{dl}{dt} \quad \text{So} \quad \frac{1}{12l} \frac{dS}{dt} = \frac{dl}{dt}
\]

\[
\frac{dV}{dt} = 3l^2 \frac{dl}{dt} = \frac{3 \times 5^2}{12} \frac{dS}{dt} = \frac{75}{4} \frac{dS}{dt}
\]

\[
\left. \frac{dV}{dt} \right|_{l=5} = \frac{75}{4} \times 1000 = 18750 \text{ in}^3/\text{sec}
\]

8. Find the points on the curve \( y^2 + 2x = 9 \) which are closest to the point \((0,0)\).

Let \((x,y)\) be a point on the curve \(D\) be the distance from \((x,y)\) to \((0,0)\).

\[
D = \sqrt{x^2 + y^2} = \sqrt{x^2 + 9 - 2x}
\]

\[
\frac{dD}{dx} = \frac{2x - 2}{2x^2 + 9 - 2x}
\]

\[
D' = -\frac{1}{x^2 + 9 - 2x}
\]

The minima occurs when \( x = 1 \)

The closest points are \((1, \sqrt{7})\), \((1, -\sqrt{7})\)
9. Solve the Initial Value Problem \( \frac{dy}{dx} = x^2 y^2 \), \( y(2) = 1 \).

\[
\int \frac{dy}{y^2} = \int dx
\]

\[
-\frac{1}{y} = \frac{x^4}{4} + C
\]

\[
-1 = 4 + C
\]

\[
-5 = C
\]

\[
-\frac{1}{\frac{x^4}{4} - 5} = y
\]

10. Let \( f(x) = x^2 + x \). Simplify the expression \( \sum_{i=1}^{n} f\left( \frac{3i}{n} \right) \). Your answer is not allowed to have a summation sign or ...

\[
\sum_{i=1}^{n} \left[ \left( \frac{3i}{n} \right)^2 + \frac{3i}{n} \right] = \frac{9}{n^2} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} i = \frac{9}{2} \frac{n(n+1)(2n+1)}{6} + \frac{3}{2} \frac{n(n+1)}{2}
\]
11. Find the exact amount of area inside the following 50 boxes. The base of each box has the same size.

\[
\text{Area} = \frac{1}{50} \cdot \frac{1}{50} + \frac{1}{50} \left( \frac{2}{50} \right)^2 + \frac{1}{50} \left( \frac{3}{50} \right)^2 + \ldots + \frac{1}{50} \left( \frac{50}{50} \right)^2 \\
= \left( \frac{1}{50} \right)^3 \left( 1 + 2^2 + 3^2 + \ldots + 50^2 \right) \\
= \left( \frac{1}{50} \right)^3 \frac{(50)(51)(101)}{6}
\]
12. Find the area of region between $x - 2y = 0$ and $y^2 - 2x = 0$.

\[
= \frac{1}{2} y^2
\]

\[
(8, 4)
\]

\[
(0, 0)
\]

Intersection:

\[
\begin{align*}
& y^2 - 4y = 0 \\
& y = 0, 4
\end{align*}
\]

\[
\int_0^4 2y - \frac{1}{2} y^2 \, dy = \left[ y^2 - \frac{1}{6} y^3 \right]_0^4 = 16 - \frac{32}{3} = \frac{16}{3}
\]

13. Find the volume of the solid which is obtained by revolving the region of problem 12 about the $x$-axis.

Spin two rectagular, get a shell or

\[
\text{Vol} = 2\pi \int_0^4 y \left( 2y - \frac{1}{2} y^2 \right) \, dy = 2\pi \int_0^4 2y^2 - \frac{1}{2} y^3 \, dy
\]

\[
= 2\pi \left[ \frac{2}{3} y^3 - \frac{1}{8} y^4 \right]_0^4 = 2\pi \cdot 64 \left[ \frac{8}{3} - \frac{1}{2} \right] = 2\pi \cdot 64 \left( \frac{1}{6} \right) = \frac{64\pi}{3}
\]