The purpose of these talks is to prove (some parts of) the following result.

**Theorem.** Let $k$ be a field of positive characteristic $p$, $P$ be the polynomial ring $k[x_1, \ldots, x_n]$, $C$ be the homogeneous complete intersection ideal $C = (f_1, \ldots, f_m)$ in $P$ and $R$ be $P/C$. Let $I$ be a homogeneous ideal in $P$ with $P/I$ a finite dimensional vector space over $k$. Suppose that the socle degrees of $R/I$ are $d_1 \leq \cdots \leq d_\ell$ and that the socle degrees of $R/I^pR$ are $D_1 \leq \cdots \leq D_L$. Then the following statements are equivalent:

1. $L = \ell$ and $D_i = pd_i - (p - 1)a(R)$ for all $i$, and
2. The ring $R/I$ has finite projective dimension as an $R$-module.

**Remark.** In the present context $a(R)$ is $\sum |f_i| - \sum |x_i|$.

**Proof of (1) \iff (2) when $C = 0.$** The ring $P$ is regular. Every $P$-module has a finite resolution by free $P$-modules. Let

$$\mathbb{F} : 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$$

be the minimal homogeneous resolution of $P/I$ by free $P$-modules, with $F_n = \bigoplus_i P(-b_i)$. There are two ingredients to the proof.

(A) The number of back twists in $\mathbb{F}$ is exactly equal to the dimension of the socle of $P/I$; furthermore, $b_i$ and $d_i = b_i + a(P)$.

(B) One obtains the minimal free resolution of $P/I^p$ by applying the Frobenius functor to $\mathbb{F}$.

As soon as you buy (A) and (B), then the proof is complete. Ingredient (B) tells us that the back twists in the $P$-resolution of $P/I$ are $pb_i$, with $1 \leq i \leq \ell$. Thus, by (A):

$$D_i = pb_i + a(P) = p(b_i + a(P)) - (p - 1)a(P) = pd_i - (p - 1)a(P).$$

**A quick illustration.** Let $P = k[x, y]$ and $I = (x^2, xy, y^2)$. The $P$-resolution of $P/I$ is

$$0 \to P(-3)^2 \to P(-2)^3 \to P, \quad \begin{bmatrix} x & 0 \\ y & x \\ 0 & y \end{bmatrix}, \quad \begin{bmatrix} y^2 & -xy & x^2 \end{bmatrix}, P.$$
The resolution of $P/I^{[p]}$ is

$$0 \to P(-3p)^2 \xrightarrow{egin{bmatrix} x^p & 0 \\ y^p & x^p \\ 0 & y^p \end{bmatrix}} P(-2p)^3 \xrightarrow{egin{bmatrix} y^{2p} & -x^py^p & x^{2p} \end{bmatrix}} P.$$

We have $a(P) = -2$. We saw that $x$ and $y$ form a basis for the socle of $P/I$. So the socle degrees of $P/I$ are $d_1 = 1 \leq d_2 = 1$. The back twists in the resolution of $P/I$ are $b_1 = 3 \leq b_2 = 3$. We see that $3 - 2 = 1$, so $b_i + a(P) = d_i$. We also see that $x^{p-1}y^{2p-1}$ and $x^{2p-1}y^{p-1}$ are in the socle of $P/I^{[p]}$. One can show that $x^{p-1}y^{2p-1}$ and $x^{2p-1}y^{p-1}$ are a basis for the socle of $P/I^{[p]}$. So the socle degrees of $P/I^{[p]}$ are $D_1 = 3p - 2 \leq D_2 = 3p - 2$; the back twists in the resolution of $P/I^{[p]}$ are $B_1 = 3p \leq B_2 = 3p$; and $a(P)$ is still $-2$. We have $D_i = B_i + a(P)$ and also $D_i = pd_i - (p - 1)a(P)$, for both $i$.

Ingredient (B), in the present form, is due to Kunz (1969) – this is the paper that got commutative algebraists (especially Peskine, Spzio, Hochster) using Frobenius methods. One could also think of this assertion as an application of “What makes a complex exact?” (John Olmo lectured on this last Fall). The complex $\mathbb{F}$ is a resolution, so the ranks of its matrices behave correctly and the grade of the ideals of matrix minors grow correctly. If one raises each entry of each matrix to the $p$th power, then the ranks of the new matrices are the same as the ranks of the old matrices (since $\det M^{[p]} = (\det M)^p$ because the characteristic of the ring is $p$), and the grade of the ideals of minors also remains unchanged!

I will give two explanations for ingredient (A). The quick argument is that one may commute $\text{Tor}_n^P(P/I, k)$ using either coordinate. If one resolves $P/I$, then applies $\_ \otimes_P k$, and then computes homology, then one sees that

$$\text{Tor}_n^P(P/I, k) = \bigoplus_i k(-b_i).$$

In other words, the generators of Tor have degrees $b_1 \leq \ldots$. On the other hand, if one resolves $k$, then applies $P/I \otimes_P \_ \otimes_P k$, and then computes homology, then one sees that

$$\text{Tor}_n^P(P/I, k) = \bigoplus_i \frac{I: \mathfrak{m}^i}{I}(a(P)).$$

In other words, the the generators of Tor have degrees $d_1 - a(P), \ldots$ (where the socle degrees of $P/I$ are $d_1, \ldots$). So $d_i = b_i + a(P)$ as claimed.
My second argument is exactly the same as my first, except, instead of stating the abstract result that Tor may computed in either coordinate, I reprove this result, giving a construction which associates an element of the socle of $P/I$ to each basis vector at the back of the resolution of $P/I$. The constructive argument takes longer, but shows what is really happening. Let $\mathbb{F}$ be a resolution of $P/I$, as above. Let $\mathbb{G}$ be the Koszul complex which resolves $k$. One can directly show that there is an isomorphism

\[(*) \quad H_n(\mathbb{F} \otimes k) \cong H_n(\text{Tot}(\mathbb{F} \otimes \mathbb{G})) \cong H_n(P/I \otimes \mathbb{G}).\]

Anyway, I think that the best way to convey the idea of $(*)$ is to work out the example where $I = (x^2, xy, y^2)$. In this case, $\mathbb{F}$ is

\[
\begin{align*}
0 \to P(-3)^2 & \to P(-2)^3 & \to P, \\
& \mathbb{F}_2 & \mathbb{F}_1 & \mathbb{F}_0 \\
& f_2 = \begin{bmatrix} x & 0 \\ y & x \\ 0 & y \end{bmatrix} & f_1 = \begin{bmatrix} y^2 & -xy & x^2 \end{bmatrix}
\end{align*}
\]

$\mathbb{G}$ is

\[
\begin{align*}
0 \to P(-2) & \to P(-1)^2 & \to P, \\
& \mathbb{G}_2 & \mathbb{G}_1 & \mathbb{G}_0 \\
& g_2 = \begin{bmatrix} y \\ -x \end{bmatrix} & g_1 = \begin{bmatrix} x & y \end{bmatrix}
\end{align*}
\]
and $\mathbb{F} \otimes \mathbb{G}$ is

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & & & \\
0 & \overset{f_2 \otimes 1}{\longrightarrow} & F_1 \otimes G_2 & \overset{f_1 \otimes 1}{\longrightarrow} F_0 \otimes G_2 \\
\downarrow 1 \otimes g_2 & & \downarrow 1 \otimes g_2 & \downarrow 1 \otimes g_2 \\
0 & \overset{f_2 \otimes 1}{\longrightarrow} & F_1 \otimes G_1 & \overset{f_1 \otimes 1}{\longrightarrow} F_0 \otimes G_1 \\
\downarrow 1 \otimes g_1 & & \downarrow 1 \otimes g_1 & \downarrow 1 \otimes g_1 \\
0 & \overset{f_2 \otimes 1}{\longrightarrow} & F_1 \otimes G_0 & \overset{f_1 \otimes 1}{\longrightarrow} F_0 \otimes G_0.
\end{array}
\]

Start with $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes 1$ in $F_2 \otimes G_0$ in the lower left hand corner. We see that this element represents an element of the homology of $H_2(\mathbb{F} \otimes k)$. One can extend this element to get an element of the homology of $H_2(\text{Tot}(\mathbb{F} \otimes \mathbb{G}))$:

\[
1 \otimes y
\]

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow 1 \otimes \begin{bmatrix} y^2 \\ -xy \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}
\]

The indicated element of $H_2(\text{Tot}(\mathbb{F} \otimes \mathbb{G}))$ gives rise to the element $y$ of the socle of $P/I$. To answer the question that our freshman ask: "Yes, it always works like that." We can use the idea of the snaky game to prove both isomorphisms in (*).

**Now we work on (1) $\implies$ (2).** We want to prove that $\text{pd}_R(R/I R) < \infty$. We apply the Theorem of Avramov and Claudia Miller (see the last seminar talk given by John Olmo last semester.) It suffices to prove that $\text{Tor}_R^1(R/I R, \mathcal{C}) = 0$. In other words, it suffices to show that if

\[
\begin{array}{c}
R^{b_2} \overset{d_2}{\longrightarrow} R^{b_1} \overset{d_1}{\longrightarrow} R \rightarrow R/I \rightarrow 0
\end{array}
\]

is exact

\[
\begin{array}{c}
R^{b_2} \overset{d_2^{[p]}}{\longrightarrow} R^{b_1} \overset{d_1^{[p]}}{\longrightarrow} R \rightarrow R/\mathcal{I}^{[p]} \rightarrow 0
\end{array}
\]

is exact.
In other words, it suffices to show that

\[ (***) \quad I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]}C. \]

I will show that (***) implies (**). (This is a rather grubby calculation. I do it to show that our goal is very concrete! One can read the calculation backwards to show that (**) implies (***)).

I make my calculation at the \( P \)-level. Let \( a_1, \ldots, a_{b_1} \) generate \( I \) in \( P \); so,

\[ d_1 = [a_1 \ldots a_{b_1}] \]

and

\[ d_1^{[p]} = [a_1^p \ldots a_{b_1}^p]. \]

We think of \( d_2 \) as having two pieces:

\[ d_2 = [d'_2 \quad d''_2] \]

where

\[ P^{b_2'} \xrightarrow{d'_2} P^{b_1} \xrightarrow{d_1} P \]

is exact (and \( d''_2 \) is all of the extra columns that describe elements of \( I \) which are also in \( C \).) Recall that Kunz’s Theorem (ingredient (B) of the other direction) tells us that

\[ P^{b_2'} \xrightarrow{(d'_2)^{[p]}} P^{b_1} \xrightarrow{d^{[p]}_1} P \]

is exact.

Suppose \( v \) is in \( P^{b_1} \) with \( d_1^{[p]}(v) \in C \). In other words,

\[ d_1^{[p]}(v) \in I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]}C. \]

So, there exist \( s_1, \ldots, s_t \in I \cap C; \alpha_1, \ldots, \alpha_t \) in \( P \); and \( c_1, \ldots c_{b_1} \) in \( C \) so that

\[ d_1^{[p]}(v) = \sum_{i=1}^{t} \alpha_i s_i^p + \sum_{i=1}^{b_1} a_i^p c_i. \]

Of course, there exists \( v_i \in P^{b_1} \) with \( d_1(v_i) = s_i \) (and therefore also \( d_1^{[p]} v_i^{[p]} = s_i^p \)).

So,

\[ d_1^{[p]}(v) = d_1^{[p]} \left( \sum_{i=1}^{t} \alpha_i v_i^{[p]} + \begin{bmatrix} c_1 \\ \vdots \\ c_{b_1} \end{bmatrix} \right). \]
So,

\[ v - \sum_{i=1}^{t} \alpha_i v_i[p] = \begin{bmatrix} c_1 \\ \vdots \\ c_{b_1} \end{bmatrix} \]

is killed by \( d_1[p] \); hence is in the image of \( (d_2')[p] \). Finally, \( d_1(v_i) = s_i \in I \cap C \), so \( v_i = d_2''(w_i) \) for some \( w_i \); hence, \( v_i[p] = (d_2'')[p](w_i[p]) \). Thus,

\[ v \in \text{im} d_2'[p] + CP^{b_1}, \]

as desired.

To be continued ...