SOCLE DEGREES, RESOLUTIONS, AND FROBENIUS POWERS

The set up.
• $k$ is a field of characteristic $p > 0$.
• $R$ is a graded ring over $k$.
• $m$ is the maximal homogeneous ideal of $R$.
• $J$ is a homogeneous $m$-primary ideal of $R$.

The question.
• Adela asked “How are the socle degrees of $R/J[q]$ related to the socle degrees of $R/J$?”

The notation of the question.
• The socle of the ring $S$ is \( \{ s \in S \mid m_S s = 0 \} \).
• $q = p^e$ for some exponent $e$.
• $J[q] = (\{ j^q \mid j \in J \})$.

Example. We calculate the socle degrees of $R/J[p^e]$ for for $R = \mathbb{Z}/2[x, y, z]/(f)$, where $f = x^5 + y^5 + z^5$ and $J = (x, y, z)$. We learn

<table>
<thead>
<tr>
<th>$e$</th>
<th>socle degrees</th>
<th>socle basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0:1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3:1</td>
<td>$xyz$</td>
</tr>
<tr>
<td>2</td>
<td>9:1</td>
<td>$x^3y^3z^3$</td>
</tr>
<tr>
<td>3</td>
<td>12:1 16:1</td>
<td>$x^4y^4z^4, x^2y^7z^7$</td>
</tr>
<tr>
<td>4</td>
<td>22:1 30:1</td>
<td>$x^4y^{14}z^4 + x^4y^9z^9 + x^4y^4z^{14}, y^{15}z^{15}$</td>
</tr>
<tr>
<td>5</td>
<td>42:1 58:1</td>
<td>$x^4y^{29}z^9 + x^4y^{19}z^{19} + x^4y^9z^{29}, xy^{26}z^{31}$</td>
</tr>
</tbody>
</table>

After a while: if the socle degrees of $R/J[q]$ are \( \{d_i\} \), then the socle degree of $R/J[pq]$ are \( \{pd_i - (p - 1)2\} \).
**Folklore.** If $\text{pd}_R R/J < \infty$, then the socles of $R/J$ and $R/J[q]$ have the same dimension and if the socle degrees of $R/J$ are $d_1 \leq \cdots \leq d_s$ then the socle degrees of $R/J[q]$ are $D_1 \leq \cdots \leq D_s$ with $D_i = q d_i - (q - 1) a$.

*Reason.* In the above notation, the generator degrees of the canonical module $\omega$ of $R/J$ are

$$-d_s \leq \cdots \leq -d_1.$$ 

The canonical module is

$$\text{Ext}^\text{top}_R (R/J, \omega_R),$$

and $\omega_R = R(a(R))$ if $R$ is Gorenstein; thus, the degrees of the generators of $\omega$ are given by the back twists in the $R$-resolution $\mathcal{F}$ of $R/J$. The resolution of $R/J[q]$ is $\mathcal{F}[q]$.

**Theorem [K,V].** If $R$ is a complete intersection, then the converse of folklore is true.

**Moral.**
1. At least sometimes, if you know the socle degrees of $R/J$, then you know the graded betti numbers in the tail of the resolution of $R/J$.
2. At least sometimes, if the socle degrees grow “correctly” as you apply the Frobenius homomorphism, then the tail of the resolution of $R/J[q^e]$ is independent of $e$. 
Example. Adela and I found other examples in which the numbers made it look like the tail of the resolution of $R/J^{[p^e]}$ is independent of $e$

Let $P$ be the polynomial ring $\frac{\mathbb{Z}}{11} [x, y, z]$, $f$ be the element $x^3 + y^3 + z^3$ of $P$, $R$ be the hypersurface ring $P/(f)$, and $J$ be the ideal $(x^5, y^5, z^5)$ of $R$. The graded betti numbers in the $R$-resolution of $R/J^{[p^e]}$ are:

$$\cdots \to R(-9)^1 \oplus R(-10)^3 \to R(-8)^3 \to R(-5)^3 \to R \to R/J^{[5^0]} \to 0.$$  

$$\cdots \to R(-39)^1 \oplus R(-40)^3 \to R(-38)^3 \to R(-25)^3 \to R \to R/J^{[5^1]} \to 0.$$  

$$\cdots \to R(-189)^1 \oplus R(-190)^3 \to R(-188)^3 \to R(-125)^3 \to R \to R/J^{[5^2]} \to 0.$$  

$$\cdots \to R(-939)^1 \oplus R(-940)^3 \to R(-938)^3 \to R(-625)^3 \to R \to R/J^{[5^3]} \to 0.$$  

$$\cdots \to R(-4689)^1 \oplus R(-4690)^3 \to R(-4688)^3 \to R(-3125)^3 \to R \to R/J^{[5^4]} \to 0.$$  

It looks like there is a resolution

$$\mathbb{F}: \cdots \to R(-1)^1 \oplus R(-2)^3 \to R \to R/J^{[5^e]} \to 0,$$

which is independent of $e$ so that for each $e$ there exists $t_e$ so that the resolution of $R/J^{[p^e]}$ is

$$\mathbb{F}(-t_e) \to R(-5^{e+1}) \to R \to R/J^{[5^e]} \to 0.$$
In these examples I did row and column operations to the matrix in position 3. Each matrix can be transformed into

\[
\begin{bmatrix}
0 & -x^2 & -y^2 & -2z \\
x^2 & 0 & -z^2 & 2y \\
y^2 & z^2 & 0 & -2x \\
2z & -2y & 2x & 0
\end{bmatrix}.
\]

The purpose of my talk.

1. I will show a situation where the graded betti numbers in the tail of the resolution of \( R/J \) are completely determined the socle degrees of \( R/J \).
2. I will apply apply (1) twice and obtain a situation where the tail of the resolution of \( R/J^{[q]} \) is a shift of the tail of the resolution of \( R/J \) as a graded module - I make no claim about the differential at this point.

**Theorem [K,U].** Let \( P = k[x,y,z] \), \( f \in P \) homogeneous, \( R = P/(f) \), and \( a = a(R) = |f| - 3 \). Let \( I \) be a homogeneous grade three Gorenstein ideal in \( P \), \( b_0 \) be the back twist in the \( P \)-resolution of \( P \), and \( J = IR \). Let

\[
F_{0,\cdot} : \ldots \xrightarrow{d_{0,\cdot}} F_{0,3} \xrightarrow{d_{0,2}} F_{0,2} \xrightarrow{d_{0,1}} F_{0,1} \xrightarrow{d_{0,0}} R \to R/J \to 0
\]

be the graded minimal R-resolution of \( R/J \), and \( \{\sigma_{0,i} \mid 1 \leq i \leq s_0\} \) be the socle degrees of \( R/J \). Assume

(1) \( \mu(I) = \mu(J) \),
(2) \( \text{rank} \ F_{0,2} = \dim_k \text{soc} \frac{R}{J} \), and
(3) \( \sigma_{0,i} + \sigma_{0,j} \neq b_0 + 2a \) for any pair \( (i,j) \). Then

\[
F_{0,2} = \bigoplus_{i=1}^{s_0} R(-(b_0 + a - \sigma_{0,i})),
\]

\[
F_{0,3} = \bigoplus_{i=1}^{s_0} R(-(\sigma_{0,i} + 3)), \text{ and}
\]

\[
F_{0,i+2} = F_{0,i}(-|f|).
\]

**Corollary [K,U].** Assume all of the above and that \( \mu(J^{[q]}) = \mu(I) \). If \( \text{soc} R/J^{[q]} = \text{soc} R/J \left[ -\frac{b_0(q-1)}{2} \right] \), then

\[
F_{e,i} = F_{0,i} \left[ -\frac{b_0(q-1)}{2} \right], \quad \forall i \geq 2.
\]

**Proof of Corollary.** The Corollary follows quickly from the Theorem. Make sure that all of the hypothesis apply to \( J \) and \( J^{[q]} \).
Example. In the earlier example, $b_0 = 15$, $a = 0$, and the shift from $J$ to $J^{[p^e]}$ is $\frac{15(5^e-1)}{2}$ and this is 30, 180, 930, and 4680 for $e$ equal to 1, 2, 3, and 4.

Outline of the proof of the Theorem. Let $Z = \text{im} d_{0,2}$. There are three parts to the proof.

**Part 1.** There exists $Z' \subset Z$ such that

$$\omega(-b_0 - a) \cong \frac{I : f}{I}(-|f|) \cong \frac{Z}{Z'}$$

- Knowledge of the generator degrees of $\omega$ is equivalent to knowledge of the socle degrees.
- The generators of $Z$ have the same degrees as the generators of $F_{0,2}$.
- The hypothesis $\text{rank } F_{0,2} = \dim_k \text{soc } R_J$ tells us that $Z$ and $\frac{Z}{Z'}$ have the same generators.
- This finishes the $F_{0,2}$ part of the argument.

**Part 2.** Eisenbud proved that if $R = P/(f)$ is a hypersurface ring and $M$ is a maximal Cohen-Macaulay module over with no free summands, then $M$ has a periodic resolution of period two given by a matrix factorization of “$f$”. Our $Z$ is $Z_{\text{periodic}} \oplus Z_{\text{free}}$. The maps $F_{0,3} \to F_{0,2} \to Z$ decompose as

$$F_{0,3} \xrightarrow{d_{0,3,\text{periodic}}} F_{0,2,\text{periodic}} \oplus F_{0,2,\text{free}} \xrightarrow{d_{0,2,\text{periodic}} 0} Z_{\text{periodic}} \oplus Z_{\text{free}}.$$  

- Now one makes 2 fairly easy homological calculations:

$$F_{0,3}^*(|f|) \xrightarrow{Z^*(a) \to \omega} (Z_{\text{periodic}})^* \text{ and } (Z_{\text{periodic}})^* \text{ have the same generator degrees}$$

- At this point we know

$$\mu(Z^*) = \mu((Z_{\text{periodic}})^*) + \mu((Z_{\text{free}})^*) = \text{rank } F_{0,2,\text{periodic}} + \text{rank } F_{0,2,\text{free}} = \text{rank } F_{0,2} = \mu(\omega).$$
• So, $Z^*(a)$ and $\omega$ have the same generator degrees.
• As soon as we show that $Z_{\text{periodic}} = Z$, then we know the the relationship between the generator degrees of $\mathbb{F}_{0,3}^*$ and the generator degrees of $\omega$. This completes the proof of the Theorem.

Part 3. If $\mathfrak{z}$ generates a free summand of $Z$, then
• the degree of the corresponding element in $Z^*(a)$ is a generator degree of $\omega$, and
• the degree of $\mathfrak{z}$ is a generator degree of $\omega(-b_0 - a)$; however,
• the hypothesis $\sigma_{0,i} + \sigma_{0,j} \neq b_0 + 2a$ for any pair $(i, j)$ prohibits the existence of such a $\mathfrak{z}$.

• We prove the homological assertions of Part 2.
  • We produce $Z^*(a) \twoheadrightarrow \omega$.

  The surjection $R \twoheadrightarrow R/J$ tells me that

  $$\omega_{R/J} = \text{Ext}^\text{dim}_R R_{R/J}(\omega_{R}) = \text{Ext}^2_R(R/J, R(a)) = \text{Ext}^1_R(J, R(a)).$$

  Apply $\text{Hom}_R(\_, R(a))$ to

  $$0 \rightarrow Z \rightarrow \mathbb{F}_{0,1} \rightarrow J \rightarrow 0$$

  to get

  $$0 \rightarrow J^*(a) \rightarrow \mathbb{F}_{0,1}^*(a) \rightarrow Z^*(a) \rightarrow \text{Ext}^1_R(J, R(a)) \rightarrow 0.$$

  • We prove that $\mathbb{F}_{0,3}^*([f])$ and $(Z_{\text{periodic}})^*$ have the same generator degrees.
    • There are two steps. The first is routine. Apply $\text{Hom}(\_, R)$ to the exact sequence
      $$\mathbb{F}_{0,3} \xrightarrow{d_{0,3,\text{periodic}}} \mathbb{F}_{0,2,\text{periodic}} \rightarrow Z_{\text{periodic}} \rightarrow 0$$

      to see that $(Z_{\text{periodic}})^* = \text{ker}(d_{0,3,\text{periodic}}^*)$.
    
    • The other step is sneaky. Extend the periodic resolution one step to the right:

      $$\mathbb{F}_{0,4} \xrightarrow{d_{0,4}} \mathbb{F}_{0,3} \xrightarrow{d_{0,3,\text{periodic}}} \mathbb{F}_{0,2,\text{periodic}} \rightarrow \mathbb{F}_{0,3}([f]) \xrightarrow{d_{0,3,\text{periodic}}([f])} \mathbb{F}_{0,2,\text{periodic}}([f]) \rightarrow Z_{\text{periodic}}([f]) \rightarrow 0.$$
The module $Z_{\text{periodic}}$ is a maximal Cohen-Macaulay module; so, 
$\text{Ext}_R^i(Z_{\text{periodic}}, R) = 0$ for all positive $i$; hence,

$0 \rightarrow (Z_{\text{periodic}}(|f|))^* \rightarrow (F_0,2,\text{periodic}(|F|))^* \rightarrow (F_0,3,\text{periodic}(|f|))^* \rightarrow (F_0,3)^*$

is exact and

$$(Z_{\text{periodic}})^* = \ker(d_{0,3,\text{periodic}}^*) = \frac{(F_0,3(|f|))^*}{\text{im}((d_{0,3,\text{periodic}}(|f|))^*)}.$$

- **We prove the assertions of Part 1.**

  - We connect $\omega$ and $\frac{I:f}{I}$.

    The surjection $P/I \rightarrow R/J$ gives

    $$\omega_{R/J} = \text{Ext}^{\dim P/I - \dim R/J}(R/J, \omega_{P/I}) = \text{Hom}(P/(I,f), P/I(a(P/I)) = \frac{I:f}{I}(b_0 - 3).$$

  - We connect $\frac{I:f}{I}$ and $Z$.

    Let $d_{1,0} = [\bar{g}_1, \ldots, \bar{g}_n]$, where $(g_1, \ldots, g_n)$ is a minimal generating set for $I$ in $P$. Of course, $Z$ is the kernel of $d_{1,0}$. If $u \in I:f$, then $uf = \sum_{i=1}^n A_i g_i$ for some $A_i$ in $P$. The association

    $$u \mapsto \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_n \end{bmatrix}$$

    induces an isomorphism

    $$\frac{I:f}{I} (-|f|) \rightarrow Z/Z',$$

    where $Z'$ is the submodule of $Z$ which comes from relations on $[g_1, \ldots, g_n]$ in $P$. $\square$

**One Final Remark.** The isomorphism $\frac{Z}{Z'} \cong \omega(-b_0 - a)$ shows that

$$\dim \text{soc} \ R/J \leq \text{rank} \ F_{0,2}$$

automatically happens and equality occurs if and only if $Z' \subseteq mZ$. If $Z' \subseteq mZ$ occurs at $J$, then the corresponding statement for $J^{[q]}$ is even more true. This explains why we did not include

$$\dim \text{soc} \ R/J^{[q]} = \text{rank} \ F_{e,2}$$

as a hypothesis in the Corollary.