Abstract

In 2005 Cameron and Walker classified all finite simple graphs \( G \) such that the matching number of \( G, m(G) \), is equal to the induced matching number of \( G, i(G) \). We call such graphs Cameron-Walker graphs. This class of graphs is of particular interest as these graph theoretic invariants provide upper and lower bounds for the Castelnuovo-Mumford regularity of the ring \( R/I(G) \), where \( R \) is the polynomial ring in \( |V(G)| \) variables and \( I(G) \) is the edge ideal of \( G \). Here we explore other algebraic properties of the edge ideals of Cameron-Walker graphs such as (sequentially) Cohen-Macaulayness, (pure) shellability, and (pure) vertex decomposability.

Introduction and Preliminary Definitions

Let \( G \) be a finite simple graph on vertex set \( [n] := \{1, \ldots, n\} \) with edge set \( E(G) \).

**Definition.** The edge ideal of \( G \) is then defined to be the ideal
\[
I(G) = \langle x_{ij} \mid (i, j) \in E(G) \rangle \subseteq \mathbb{K}[x_1, \ldots, x_n]
\]
where \( \mathbb{K} \) is a field.

Since \( I(G) \) is a square-free monomial ideal, we can realize \( I(G) \) as the Stanley-Reiser ideal of the independence complex of \( G \), denoted \( \Delta(G) \).

**Definition.** Let \( G \) be a finite simple graph on vertex set \( [n] := \{1, \ldots, n\} \) with edge set \( E(G) \).

1. A set \( I \subseteq \{1, \ldots, n\} \) is independent in \( G \) if \( (i, j) \notin E(G) \) for all \( i, j \in I \).
2. A set \( I \subseteq \{1, \ldots, n\} \) is a maximal independent set in \( G \).
3. A set \( I \subseteq \{1, \ldots, n\} \) is a minimal independent set in \( G \).

**Definition.** A simplicial complex \( \Delta \) is shellable if all of its facets (maximal faces) can be listed \( F_1, \ldots, F_t \) in such a way that
\[
\bigcap_{j=1}^{t} F_j = \bigcup_{j=1}^{t} F_j
\]
is a pure simplicial complex of dimension \( d(F_i) - 1 \) for all \( 1 < i \leq t \).

**Definition.** A module \( M \) is sequentially Cohen-Macaulay if there exists a filtration of \( M \)
\[
0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{i-1} \subseteq M_{i} \subseteq \cdots \subseteq M_t = M
\]
such that \( M_i/M_{i-1} \) is Cohen-Macaulay for all \( 1 \leq i \leq r \).

**What is Known**

**bipartite**

We say a graph \( G \) is bipartite if there is a partition of the vertex set \( V(G) = \{a_i \cup [m]\} \) such that for all edges \( (i, j) \in E(G) \) we have \( i \in [n] \) and \( j \in [m] \).

- [Ravinda; Villarreal 2007] Unmixed bipartite graphs are Cohen-Macaulay.
- [Van Tuyl 2009] Sequentially Cohen-Macaulay bipartite graphs are vertex decomposable, decomposable.

**chordal**

We say a graph \( G \) is chordal if every cycle of length at least 4 has a chord.

- [Francisco-Van Tuyl 2007] Chordal graphs are sequentially Cohen-Macaulay.
- [Dochtermann-Enstrom 2009] Chordal graphs are vertex decomposable.
- [Woodroofe 2006] Graphs with no chordless cycle of length other than 3 or 5 are vertex decomposable.

**Cameron-Walker Graphs**

**Definition.** A set \( M \subseteq E(G) \) is said to be a matching if for all \( e, e' \in E(G) \) with \( e \neq e' \), we have \( e \cap e' = \emptyset \). The matching number, denoted \( m(G) \), is defined to be
\[
m(G) := \max \{ |M| \mid M \text{ is a matching in } G \}.
\]

**Definition.** A set \( M \subseteq E(G) \) is said to be an induced matching if for all \( e, e' \in E(G) \) with \( e \neq e' \), there does not exist \( f \in E(G) \) such that \( e \cup f \neq \emptyset \) and \( e' \cup f \neq \emptyset \). The induced matching number, denoted \( i(G) \), is defined to be
\[
i(G) := \max \{ |M| \mid M \text{ is an induced matching in } G \}.
\]

**Example.** For \( G \) in the first example, \( \{2, 3, 4, 5\} \) is a matching but not an induced matching and we have \( m(G) = 2, i(G) = 1 \).

These graph theoretic invariants are of particular interest to algebraists as they provide bounds for the Castelnuovo-Mumford regularity of \( S/I(G) \).

**Theorem.** [Katzman '05, Ha-Van Tuyl '07] Given a finite simple graph \( G \) with edge ideal \( I(G) \subseteq R \subseteq \mathbb{K} \), we have the following inequality
\[
i(G) \leq \text{reg } S/I(G) \leq m(G).
\]

**Theorem.** [Cameron-Walker '05] Suppose \( G \) is a finite simple graph such that \( i(G) = m(G) \). Then \( G \) is one of the following:
1. the empty graph, i.e. \( E(G) = \emptyset \).
2. \( K_{n,m} \) is comprised of a bipartite graph on vertex set \( \{a_i \cup [m]\} \) such that for all \( i \in [n] \) there is at least one leaf edge connected to \( i \) and for all \( j \in [m] \) there may be a pendant triangle attached to \( j \).

**Example.** The following is an example of a Cameron-Walker graph.

**Example.** A Cameron-Macaulay Cameron-Walker graph.

It turns out we need no requirements on the number of leaves, pendant triangles, or even the structure of the supporting bipartite graph of a Cameron-Walker graph to obtain vertex decomposability.

**Theorem.** [Hibi-Higashitani-Kimura-O'Keefe '13] Every Cameron-Walker graph is vertex decomposable.

References