An example of using Representation Theory to find a resolution

I plan to give three 50 minute talks. Here is an outline of my plan.

Section 1. A brief introduction to irreducible representations of \( \text{GL}(V) \).

Section 2. A family of complexes associated to a generic map \( \varphi : F \to G \).

Section 3. The Littlewood-Richardson rule shows that the objects of section 2 are complexes.

Section 4. The Acyclicity Lemma (together with LR) yields exactness!

Section 1. A brief introduction to irreducible representations of \( \text{GL}(V) \).

Let \( K \) be a field of characteristic zero (like \( \mathbb{Q} \)) and \( V \) be a finite dimensional vector space over \( K \). There are many \( K \)-module maps \( \theta : V \to V \). However, if I restrict my attention to coordinate-free maps \( \theta \) (and this is the natural thing to do in my business – if I am trying to tell somebody a large sequence of maps (i.e., a resolution), I have to make these maps be as transparent as possible, I don’t want to have to start by telling my favorite basis!), then there are many fewer choices for \( \theta \). Indeed, in this case, \( \theta \) is multiplication by a scalar from \( K \). (This is an easy exercise in Linear Algebra. The words “\( \theta : V \to V \) is a coordinate-free map” mean that for every \( g \in \text{GL}(V) \), \( g \circ \theta \circ g^{-1} = \theta \). So, the matrix for \( \theta \), with respect to your favorite basis, commutes with all invertible matrices.)

The basic building blocks in this sequence of lectures are irreducible representations of \( \text{GL}(V) \). The first paragraph establishes \( V \) as one of these. As soon as I tell you what I am talking about, and why I care, I will tell you many more examples.

The vector space \( L \) is a representation of \( \text{GL}(V) \) (or is a \( \text{GL}(V) \)-module, or is a \( K[\text{GL}(V)] \)-module, or admits a \( \text{GL}(V) \)-action) if every change of basis in \( V \) gives rise to a corresponding change of basis in \( L \) (in a coherent manner).

Furthermore, the representation \( L \) is irreducible if the \( L \) does not contain any proper sub-representations. Notice that if \( L \) is an irreducible representation of \( \text{GL}(V) \), and \( \theta : L \to L \) is a \( K \)-linear map which is independent of the choice of basis for \( V \), (i.e., \( \theta \) is a \( \text{GL}(V) \)-equivariant map), then \( \theta \) is multiplication by a scalar. Similarly, if \( L \) and \( L' \) are non-isomorphic irreducible representations of \( \text{GL}(V) \), and \( \theta : L \to L' \) is a \( \text{GL}(V) \)-equivariant \( K \)-module homomorphism, then \( \theta \) is zero.

Remark. I believe that the last two sentences (about homomorphisms of irreducible \( \text{GL}(V) \)-modules) is known as Schur’s Lemma. For the first fifty years of my life – before I made any sense out of Representation Theory – I would tell students that Schur’s lemma said that any ring homomorphism from a field to itself is either identically zero or and isomorphism. Then I would stand back, scratch my head,
and wonder why a name was associated to such a trivial observation. Technically, I suppose, I was telling the truth in the old days. The intersection of what Schur proved and what I understood was indeed, a trivial comment. However, I am more impressed with the result now that I understand more of it. The two sentences in the preceding paragraph are the key to the whole sequence of lectures.)

**Example.** The module $\text{Sym}_d(V)$ is an irreducible $\text{GL}(V)$-representation for all non-negative integers $d$. The easiest way to deal with $\text{Sym}_d(V)$ is to pick a basis $v_1, \ldots, v_n$ for $V$. The vector space $\text{Sym}_d(V)$ is the vector space of all homogeneous polynomials of degree $d$ in the $n$ symbols $v_1, \ldots, v_n$. Do notice that $\text{Sym}_d(V)$ admits a $\text{GL}(V)$-action. (If you decide to use a new basis for $V$, you will be looking at the same set of polynomials.)

**Example.** The module $\wedge^d(V)$ is an irreducible $\text{GL}(V)$-representation for all non-negative integers $d$. My favorite basis for $\wedge^d(V)$ is

$$\{v_{i_1} \wedge \cdots \wedge v_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq n\}.$$

Once again, notice that $\wedge^d(V)$ admits a $\text{GL}(V)$-action.

**Example.** The Schur module $L_\lambda V$ is an irreducible $\text{GL}(V)$-representation for all partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. (The words “$\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition” mean $\lambda_1 \geq \cdots \geq \lambda_\ell$ are non-negative integers.) The coordinate-free definition is that $L_\lambda V$ is the image of the natural map

$$(*) \quad \wedge^{\lambda_1} V \otimes \cdots \otimes \wedge^{\lambda_\ell} V \to \text{Sym}_{\lambda_1'} V \otimes \cdots \otimes \text{Sym}_{\lambda_{\ell\text{\,last}}} V$$

where $\lambda' = (\lambda_1', \ldots, \lambda'_{\ell\text{\,last}})$ is the partition dual to $\lambda$. In particular, $L_i V = \wedge^i V$ and $L_{1\,\ell\,V} = \text{Sym}_i(V)$. 