RESOLUTIONS WHICH ARE DIFFERENTIAL GRADED ALGEBRAS
MORGANTOWN ALGEBRA DAYS
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I put a copy of this talk on my website.

1. The statement of the main result.

Theorem. *Let P be a commutative Noetherian ring and F be a resolution by finitely generated free P-modules. Assume that $F_0 = P$, F has length four, and F is self-dual, that is, $F \cong \mathrm{Hom}_P(F,P)$. Then F is a Differential Graded Algebra with Divided Powers and Poincaré Duality.*

This result is already known if P is a local Gorenstein ring and F is a minimal resolution. The purpose of the present project is to remove the unnecessary hypotheses that P is local, P is Gorenstein, and F is minimal.

2. Define the words and give an example.

First, I give an example (and simultaneously make sure that the meaning of the words is clear.) The Koszul complex is an example of a (resolution) which is a DG-algebra with divided powers and Poincaré duality.

• Let K to be a free P-module of rank n and $d : K \to P$ be a P-module homomorphism. There is a multiplication on the Koszul complex

$$0 \to \bigwedge^n K \to \cdots \to \bigwedge^2 K \to \bigwedge^1 K \to \bigwedge^0 K.$$

This multiplication respects the grading in the sense that: an element in homological position i is a sum of i-forms and if one multiplies an i-form by a j-form, one get an $(i+j)$-form in homological position $i+j$.
(In general, the multiplication in a DG-algebra resolution respects the grading.)

• The multiplication in the Koszul complex satisfies the product rule:

$$d_{i+j}(x_i \wedge x_j) = d_i(x_i) \wedge w_j + (-1)^i x_i \wedge d_j(x_j).$$

(This always happens in a DG-algebra.)

• The multiplication in the Koszul complex associates, distributes over addition, and is graded commutative:
 – homogeneous elements of even degree commute with everything, and
 – two homogeneous elements of odd degree anti-commute.
(This always happens in a DG-algebra.)

• Poincaré Duality: The multiplication in the Koszul complex in complementary degrees:

$$\bigwedge^i K \otimes \bigwedge^{n-i} K \to \bigwedge^n K$$

is a perfect pairing. (When this happens in a DG-algebra F, one says that F exhibits Poincaré Duality.)

• Divided Powers: An algebra F has Divided Powers if for each homogeneous element x (of even degree) there is a system of elements $x^{(n)}$, for $0 \le n$, such that $x^{(n)}$ behaves like $x^n/n!$ would behave if $x^n/n!$ were in F. (For an official list of axioms, look in Gulliksen and Levin, or Eisenbud, or ... .)

For the situation of this talk

$$0 \to F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0,$$

there is only one type of Divided Power appearing, namely $x_2^{(2)} \in F_4$ for $x_2 \in F_2$. Here is the point, x_2^2 is automatically 2 times some element of F_4 because

– $d_4(x_2^2) = 2d_2(x_2) \times x_2$
– $d_2(x_2) \times x_2$ is already a cycle.
– some element of F_4 is sent to $d_2(x_2) \times x_2$
– the map d_4 is an injection; so $x_2^2 = 2(\text{some element})$

If F has a DG-structure then $x_2^2 = 2(\text{some element})$ for all $x_2 \in F_2$ and this is true independent of characteristic and independent of whether 2 is a unit or not. Of course, this "some element" is called $x_2^{(2)}$.

I really like divided powers. I want to calculate explicitly the divided powers in $\bigwedge^\bullet K$, when K is a free module of rank 4. (The next few words are propaganda; but if you have not thought about it before, it might be useful propaganda.) Let $e_1, \dots, e_4$ be a basis for K. Square an arbitrary element of $\bigwedge^2 K$:

$$\Big(p_{1,2}e_1 \wedge e_2 + p_{1,3}e_1 \wedge e_3 + p_{1,4}e_1 \wedge e_4 + p_{2,3}e_2 \wedge e_3 + p_{2,4}e_2 \wedge e_4 + p_{3,4}e_3 \wedge e_4\Big)^2$$

$$= 2\Big(p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3}\Big) e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

$$= 2\,\mathrm{pf} \begin{bmatrix} 0 & p_{1,2} & p_{1,3} & p_{1,4} \\ -p_{1,2} & 0 & p_{2,3} & p_{2,4} \\ -p_{1,3} & -p_{2,3} & 0 & p_{3,4} \\ -p_{1,4} & -p_{2,4} & -p3,4 & 0 \end{bmatrix} e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

Conclude (or define)

$$\Big(p_{1,2}e_1 \wedge e_2 + p_{1,3}e_1 \wedge e_3 + p_{1,4}e_1 \wedge e_4 + p_{2,3}e_2 \wedge e_3 + p_{2,4}e_2 \wedge e_4 + p_{3,4}e_3 \wedge e_4\Big)^{(2)}$$

$$= \mathrm{pf} \begin{bmatrix} 0 & p_{1,2} & p_{1,3} & p_{1,4} \\ -p_{1,2} & 0 & p_{2,3} & p_{2,4} \\ -p_{1,3} & -p_{2,3} & 0 & p_{3,4} \\ -p_{1,4} & -p_{2,4} & -p3,4 & 0 \end{bmatrix} e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

The moral: The "right way" (i.e., painless way – the signs take care of themselves) to think about Pfaffians is by way of divided powers in an exterior algebra. In the same sense that the "right way" to think about determinants is by way of multiplication in an exterior algebra.

3. Why would one want to have DG-algebra resolutions?

1. Suppose one wants a map of complexes

$$\alpha : \bigwedge^\bullet K \to F$$

from a Koszul complex $\bigwedge^\bullet K$ to a complex F. (For example, if one wants to resolve the link $(\underline{x} : I)$, where $\underline{x}$ is a regular sequence in P, I is a perfect ideal in P, $\bigwedge^\bullet K$ resolves $P/(\underline{x})$, and F resolves P/I, then one might start with such an α.) At any rate, if F is a DG-algebra, then one need only describe a map $\alpha_1 : K \to F_1$, then one can take the rest of α to be a map of rings.

2. Let A be the ring P/I, where I is an ideal in a regular local ring $(P, \mathfrak{m}, k)$. If the minimal resolution F of A is a DG-algebra, then Avramov proved that the Eilenberg-Moore spectral sequence degenerates. When this happens many questions about the ring A may be translated into questions about the Koszul homology algebra $\mathrm{Tor}^P(A, \boldsymbol{k})$. This technique has led to the following theorems in the case when

$$(**) \quad A \text{ has small codimension or small linking number:}$$

– the Poincaré series of finitely generated A-modules have been calculated,

– the asymptotics of the Betti numbers of finitely generated A-modules has been determined,

– the Bass series of finitely generated A-modules has been found,

– (A-I-Nasseh-SW) if M and N are finitely generated A-modules and $\mathrm{Tor}_i^A(M, N) = 0$ for all large i, then M or N has finite projective dimension,

– (A-I-Nasseh-SW) if A is not Gorenstein and not an embedded deformation, then A is G-regular in the sense that every totally reflexive module over such a ring is free.

3. (This is a project with Rebecca R.G. and Adela Vraciu.) Let J be generated by a regular sequence of length four in the commutative Noetherian ring P and f be an element of P. Then to resolve $J\frac{P}{(f)}$ over $P/(f)$ it suffices

(a) to resolve $P/(J : f)$ over P,
(b) use the DG-structure on the resolution of (a), and
(c) use one more ingredient which is built out of (b).

4. Which change to the old theorem is the important change?

I promised to remove the hypotheses P is local, P is Gorenstein, and F is minimal from the old theorem.

I suspect that P is Gorenstein is not used in the original result.

I know that F is minimal is NOT needed in the old result. In the project with Rebecca and Adela, sometimes one wants to use a resolution of $P/(J:f)$ which is not a minimal resolution. I wondered if the non-minimal resolution still is a DG-algebra. The answer is yes.

An arbitrary resolution F (as described in the hypotheses of the Theorem from the beginning of the talk) over a local ring is isomorphic to a minimal resolution plus

$$0 \to \underset{\text{spot } 3}{E_1^*} \xrightarrow{\begin{bmatrix} 0 \\ \cong^* \end{bmatrix}} \underset{\text{spot } 2}{\begin{matrix} E_2 \\ \oplus \\ E_2^* \end{matrix}} \xrightarrow{\begin{bmatrix} \cong & 0 \end{bmatrix}} \underset{\text{spot } 1}{E_1} \to 0$$

One can easily extend the multiplication on the minimal resolution to a multiplication on the direct sum.

(This observation is the starting point of the present project.)

Removing the hypothesis "local" is much sneakier. The hypothesis local is used in two main spots in the original proof.

5. An outline of the proof of the Theorem, with a special emphasis on where the original hypothesis "local" had been used.

An outline of the proof follows.

Step 1. Find maps $\psi_3 : F_1 \otimes F_3 \to F_4$ and $\psi_4 : D_2F_2 \to F_4$ such that

(a) ψ_3 and ψ_4 satisfy the product rule for $0 = x_1 \cdot x_4$ and $0 = x_2 \cdot x_3$, and
(b) ψ_3 is a perfect pairing and $x_2 \mapsto \psi_4(x_2 \cdot -)$ is an isomorphism $F_2 \to \mathrm{Hom}(F_2, F_4)$

Step 2. Take ψ_3 and ψ_4 from Step 1 to be the multiplication. Make $\psi_1 : F_1 \otimes F_1 \to F_2$ do all the work. That is, figure out what ψ_1 must do in order for ψ_1, ψ_2, ψ_3, ψ_4 to be the multiplication, where ψ_2 is defined to make $y_1(x_1x_2) = (y_1x_1)x_2$:

$$\psi_3(y_1 \otimes \psi_2(x_1 \otimes x_2)) = \psi_4(\psi_1(y_1 \otimes x_1) \cdot x_2).$$

Of course, this definition makes sense because ψ_3 is a perfect pairing.

It turns out that ψ_1 must satisfy 3 hypotheses.

(a) one differential condition for $F_1 \otimes F_1 \to F_2$,
(b) one differential condition for $F_1 \otimes F_2 \to F_3$, and
(c) ψ_1 factors through $\bigwedge^2 F_1$.

Step 3. Prove that there exists a ψ_1 which satisfies all of the conditions of Step 2.

Here is how Step 3 turns out.

Step 1.(a) is easy. I will show it to you if I have time.

Step 1.(b) is obvious if P is local. (Again, I will show it to you if I have time.) It is not obvious in general. (I used the Theresa May approach.)

It is not hard to satisfy to find a ψ_1 which satisfies conditions 2.(a) and 2.(b). One then modifies ψ_1 (numerous times) to make the ultimate ψ_1 satisfy condition 2.(c). The proof in the local case is spread over two papers. The first paper proves the result when 2 is a unit (by dividing by 2). The second paper proves the result when 3 is a unit (by dividing by 3). Of course, in a local ring, either 2 is a unit or 3 is a unit. The present argument multiplies instead of dividing. It solves 2^n times 2.(a), 2.(b), and 2.(c), for some large n, and it solves 3 times 2.(a), 2.(b), and 2.(c) and then it solves the problem by taking the appropriate integral linear combination of the two solutions.

6. How to get started.

We learned the technique that is used in the proof the Buchsbaum-Eisenbud Am. J. paper. The technique is similar to the Tate method of killing cycles. One kills cycles of even degree with exterior variables and one kills cycles of odd degree with divided power variables.

The maps on the top from a complex. The maps on the bottom are a resolution. The comparison theorem yields a map of complexes from the top to the bottom. Focus on ψ_3 and ψ_4. The fact that the left most square commutes ensures that

$$0 = d_1(x_1)\cdot x_4 - \psi_3(x_1\otimes d_3(x_3)) \quad \text{and}$$

$$0 = \psi_3(d_2(x_2)\otimes x_3) + \psi_4(x_2\cdot d_3(x_3)).$$

$$\begin{tikzcd}[column sep=large]
\begin{matrix} F_2\otimes F_3 \\ \oplus \\ F_1\otimes F_4 \end{matrix} \arrow[r, "{\begin{bmatrix} 1\otimes d_3 & 0 \\ d_2\otimes 1 & -1\otimes d_4 \\ 0 & d_1\otimes 1 \end{bmatrix}}"] \arrow[d] & \begin{matrix} D_2F_2 \\ \oplus \\ F_1\otimes F_3 \\ \oplus \\ F_4 \end{matrix} \arrow[r, "{\begin{bmatrix} d_2 & -1\otimes d_3 & 0 \\ 0 & d_1\otimes 1 & d_4 \end{bmatrix}}"] \arrow[d, "{c_4=\begin{bmatrix} \psi_4 & \psi_3 & \mathrm{id}_{F_4} \end{bmatrix}}"] & \begin{matrix} F_1\otimes F_2 \\ \oplus \\ F_3 \end{matrix} \arrow[r, "{\begin{bmatrix} -1\otimes d_2 & 0 \\ d_1\otimes 1 & d_3 \end{bmatrix}}"] \arrow[d, "{c_3=\begin{bmatrix} \psi_2 & \mathrm{id}_{F_3} \end{bmatrix}}"] & \begin{matrix} \bigwedge^2 F_1 \\ \oplus \\ F_2 \end{matrix} \arrow[r, "{\begin{bmatrix} d_1 & d_2 \end{bmatrix}}"] \arrow[d, "{c_2=\begin{bmatrix} \psi_1 & \mathrm{id}_{F_2} \end{bmatrix}}"] & F_1 \arrow[r, "d_1"] \arrow[d, "="] & F_0 \arrow[d, "="] \\
0 \arrow[r] & F_4 \arrow[r, "d_4"] & F_3 \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0.
\end{tikzcd}$$

We modify ψ_3 and ψ_4 in order to make them induce the appropriate isomorphisms. (No modification is needed in the local case.) In the 2^n part of the argument, we keep ψ_3 and ψ_4, ignore the given ψ_1 and ψ_2, and build a new ψ_1 (and

ψ_2) from scratch! In the 3 part of the argument we modify the ψ_1 and ψ_2 that come from the Buchsbaum-Eisenbud-Tate technique.

7. HERE IS WHY NO MODIFICATION OF ψ_3 AND ψ_4 IS NEEDED IN THE LOCAL CASE.

Let $(-)^\vee$ denote the functor $\mathrm{Hom}_P(-,F_4)$. Define

$$\begin{tikzcd} F \arrow[d, "\Phi"] & 0 \arrow[r] & F_4 \arrow[r, "d_4"] \arrow[d, "\Phi_4"] & F_3 \arrow[r, "d_3"] \arrow[d, "\Phi_3"] & F_2 \arrow[r, "d_2"] \arrow[d, "-\Phi_2"] & F_1 \arrow[r, "d_1"] \arrow[d, "\Phi_1"] & F_0 \arrow[d, "\Phi_0"] \\ F^\vee & 0 \arrow[r] & F_0^\vee \arrow[r, "d_1^\vee"] & F_1^\vee \arrow[r, "d_2^\vee"] & F_2^\vee \arrow[r, "d_3^\vee"] & F_3^\vee \arrow[r, "d_4^\vee"] & F_4^\vee \end{tikzcd}$$

by

$$\begin{aligned} \Phi_0(x_0) &= x_0\cdot -, \\ \Phi_1(x_1) &= \psi_3(x_1\otimes -), \\ \Phi_2(x_2) &= \psi_4(x_2\cdot -), \\ \Phi_3(x_3) &= \psi_3(-\otimes x_3), \text{ and} \\ \Phi_4(x_4) &= -\cdot x_4. \end{aligned}$$

It is easy to see that Φ is a map of complexes.

If P is a local ring and F is a minimal resolution, then Φ is a map of complexes from one minimal resolution of $P/\operatorname{im} d_1$ to another minimal resolution of $P/\operatorname{im} d_1$ and Φ_0 is an isomorphism. It follows immediately that Φ is an isomorphism of complexes.