Syzygies

Graduate Colloquium
November 7, 2017

Andy Kustin

The Setup.

$\boldsymbol{k}$ is a field (For some applications $\boldsymbol{k}$ has to be infinite.)

$R = \boldsymbol{k}[x_1, \ldots, x_n]$ is a polynomial ring

M is a finitely generated graded R-module

For example, M could be an ideal I of R which is generated by homogeneous polynomials or M could be a quotient ring R/I where again I is an ideal generated by homogeneous polynomials.

If one wants to "understand" M,

one might want a minimal generating set for M. This would be a set of homogeneous elements $m_1, \ldots, m_{\beta_0}$ in M so that every element in M can be written "in terms" of $m_1, \ldots, m_{\beta_0}$ (and none of the m_i can be omitted).

In particular, every element in M has the form $\sum_{i=1}^{\beta_0} r_i m_i$ for some r_i in R.

As soon as one has a minimal generating set for M, the next natural question is "How can I tell when two elements of M are the same?"

That is, one wants to know the set

$$\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_0} \end{bmatrix} \in R^{\beta_0} \;\middle|\; \sum_{i=1}^{\beta_0} r_i m_i = 0 \right\}.$$

The set

$$\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_0} \end{bmatrix} \in R^{\beta_0} \,\middle|\, \sum_{i=1}^{\beta_0} r_i m_i = 0 \right\}$$

is another finitely generated graded R-module, called **the first syzygy module** $\mathrm{Syz}_1(M)$ of M.

Repeat the above process.

If one wants to "understand" $\mathrm{Syz}_1(M)$, one might want a minimal generating set $X_1, \ldots, X_{\beta_1}$ for $\mathrm{Syz}_1(M)$ and then one would want to know the relations on $X_1, \ldots, X_{\beta_1}$. The set of relations

$$\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_1} \end{bmatrix} \in R^{\beta_1} \,\middle|\, \sum_{i=1}^{\beta_1} r_i X_i = 0 \right\}$$

The set of relations

$$\left\{ \begin{bmatrix} r_1 \\ \vdots \\ r_{\beta_1} \end{bmatrix} \in R^{\beta_1} \;\middle|\; \sum_{i=1}^{\beta_1} r_i X_i = 0 \right\}$$

on $X_1, \ldots, X_{\beta_1}$ is called **the second syzygy module** $\mathrm{Syz}_2(M)$ of M.

One continues in this manner to find $\mathrm{Syz}_i(M)$ for all i.

The syzygy modules of M are uniquely determined up to isomorphism.

The Hilbert syzygy theorem guarantees that $\mathrm{Syz}_i(M) = 0$ for $n+1 \leq i$.

The numbers $\{\beta_i\}$ are called **the Betti numbers** of M.

The collection of free modules and induced maps

$$\cdots \to R^{\beta_2} \to R^{\beta_1} \xrightarrow{\begin{bmatrix} X_1 & \dots & X_{\beta_1} \end{bmatrix}} R^{\beta_0}$$

is called **the minimal free resolution** of M.

This might be a good time to say that in Astronomy "syzygy" refers to three celestial bodies in a straight line.

The word has Greek origin; it means "yoke".

Example Let $R = \boldsymbol{k}[x,y]$ and $M = R/(x^3, x^2y, y^2)$. Find the minimal homogeneous resolution of M by free R-modules.

It is clear that M is generated by the class $\bar{1}$ of 1 in M.

It is also clear that the multiples of $\bar{1}$ that are zero in M are $\theta \cdot \bar{1}$, where θ is in the ideal (x^3, x^2y, y^2) and that the listed generators are a minimal generating set of this ideal.

So, the ideal (x^3, x^2y, y^2) is the first syzygy module of M.

To find the second syzygy module of M, we look for

$$\left\{ \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \middle| \, r_1(x^3) + r_2(x^2y) + r_3(y^2) = 0 \in R \right\}.$$

We notice that

$$\begin{array}{cccc} y(x^3) & -x(x^2y) & & = 0 \\ & y(x^2y) & -x^2(y) & = 0 \end{array}$$

One easily shows that

$$\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ y \\ -x^2 \end{bmatrix}$$

generate the entire second syzygy module of M.

Indeed, if $r_1(x^3) + r_2(x^2y) + r_3(y^2) = 0$, then

$$x^2(r_1x + r_2(x^2y)) + r_3(y^2) = 0$$

in the UFD R. So, x^2 divides r_3. Now one is left with $r_1(x^3) + r_2(x^2y) = 0$. Again, one uses the fact that R is a UFD to show that

$$\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} \text{ is a multiple of } \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}.$$

It is obvious that there are no non-zero relations on

$$\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ y \\ -x^2 \end{bmatrix}.$$

We conclude that

$$0 \to R(-4)^2 \xrightarrow{\begin{bmatrix} y & 0 \\ -x & y \\ 0 & -x^2 \end{bmatrix}} \begin{matrix} R(-3)^2 \\ \oplus \\ R(-2)^1 \end{matrix} \xrightarrow{\begin{bmatrix} x^3 & x^2y & y^2 \end{bmatrix}} R$$

is the minimal homogeneous resolution of M. The numbers (-2), (-3), and (-4) are keeping track of the degrees.

$$0 \to R(-4)^2 \xrightarrow{\begin{bmatrix} y & 0 \\ -x & y \\ 0 & -x^2 \end{bmatrix}} \begin{matrix} R(-3)^2 \\ \oplus \\ R(-2)^1 \end{matrix} \xrightarrow{\begin{bmatrix} x^3 & x^2y & y^2 \end{bmatrix}} R$$

We can easily count $\dim_{\boldsymbol{k}} M_i$ for each i

$$\begin{aligned}
\dim_{\boldsymbol{k}} M_0 &= \dim_{\boldsymbol{k}} R_0 = 1 \\
\dim_{\boldsymbol{k}} M_1 &= \dim_{\boldsymbol{k}} R_1 = 2 \\
\dim_{\boldsymbol{k}} M_2 &= \dim_{\boldsymbol{k}} R_2 - 1 \dim_{\boldsymbol{k}} R_0 = 3 - 1 = 2 \\
\dim_{\boldsymbol{k}} M_3 &= \dim_{\boldsymbol{k}} R_3 - 1 \dim_{\boldsymbol{k}} R_1 - 2 \dim_{\boldsymbol{k}} R_0 = 4 - 2 - 2 = 0 \\
\dim_{\boldsymbol{k}} M_i &= \dim_{\boldsymbol{k}} R_i - 1 \dim_{\boldsymbol{k}} R_{i-2} - 2 \dim_{\boldsymbol{k}} R_{i-3} + 2 \dim_{\boldsymbol{k}} R_{i-4} \\
&= (i+1) - (i-1) - 2(i-2) + 2(i-3) = 0 \qquad \text{for } 4 \leq i
\end{aligned}$$

Of course, these counts are trivial for this M. As a vector space M has basis

$$1, \quad x, y \quad x^2, xy.$$

But

1. The idea that the function $i \mapsto \dim_{\boldsymbol{k}} M_i$ may be calculated from the minimal homogeneous resolution holds all the time. This function is called the **Hilbert function of** M, one could denote it $\mathrm{HF}_M(i) = \dim_{\boldsymbol{k}} M_i$.

2. The idea that there is a polynomial $\mathrm{HP}_M(i)$ that gives the Hilbert function of M for all large i holds all the time. So, $\mathrm{HP}_M(i)$ is a polynomial in i and $\mathrm{HP}_M(i) = \mathrm{HF}_M(i)$ for all large i. The polynomial $\mathrm{HP}_M(i)$ is called the **Hilbert polynomial** of M. The degree of HP_M has geometric meaning:

The degree of HP_M has geometric meaning:

$\deg \mathrm{HP}_M(i) = \mathrm{Kdim}(M) - 1$. I have used Kdim to represent **Krull dimension**. The Krull dimension of (the coordinate ring of) a finite set of points (in affine space) is zero; the Krull dimension of (the coordinate ring of) a curve (in affine space) is 1; the Krull dimension of (the coordinate ring of) a surface (in affine space) is two; the Krull dimension of $\boldsymbol{k}[x_1, \ldots, x_n]$ is n. The leading coefficient of $\mathrm{HP}_M(i)$ also has geometric meaning. This leading coefficient is $\frac{e(M)}{(\mathrm{Kdim}(M)-1)!}$. Commutative algebraists call $e(M)$ the **multiplicity of** M; Geometers call $e(M)$ the **degree of** M.

3. The idea that the "generating function" $\mathrm{HS}_M(t) = \sum_i (\dim_{\boldsymbol{k}} M_i) t^i$ (called the **Hilbert series** of M) is equal to

$$\text{a polynomial that may be read from the resolution} \cdot \mathrm{HS}_R(t)$$

$$= \frac{\text{a polynomial that may be read from the resolution}}{(1-t)^n}$$

holds all of the time.

Example. Assume that $\boldsymbol{k}$ is infinite. Consider the morphism $\Gamma : \mathbb{P}^1 \to \mathbb{P}^3$ which is given by $\Gamma([s:t]) = [s^3 : s^2t : st^2 : t^3]$. Let C be the image of Γ. Let $R = \boldsymbol{k}[x_1, x_2, x_3, x_4]$ be the homogeneous coordinate ring for $\mathbb{P}^3$.

(a) Find $I(C) = \{f \in R \mid f \text{ vanishes on } C\}$.

(b) Find the minimal homogeneous resolution of $S = R/I(C)$. (The ring S is called the *homogeneous coordinate ring of* C.) Observe that C is a curve, so the coordinate ring of some dehomogenization of C has Krull dimension 1 and S has Krull dimension 2.

(c) Compute the multiplicity of S.

(d) Does it make sense to say that the degree of C is equal to the answer to (c)?

$C = \{[s^3 : s^2t : st^2 : t^3] \in \mathbb{P}^3 \mid [s:t] \in \mathbb{P}^1\}$

(a) We see that $x_1x_3 - x_2^2$, $x_1x_4 - x_2x_3$, and $x_2x_4 - x_3^2$ all vanish on C. We see that every element of R has the form

$$\begin{aligned} &f_1(x_1,x_4) + x_2 f_2(x_1,x_4) + x_3 f_3(x_1,x_4) \\ &+\text{an element of } (x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2) \end{aligned}$$

No terms in

$$f_1(s^3,t^3) + s^2t f_2(s^3,t^3) + st^2 f_3(s^3,t^3) \tag{1}$$

can cancel; so (1) vanishes on C only if $f_1(x_1,x_4) + x_2 f_2(x_1,x_4) + x_3 f_3(x_1,x_4)$ is the zero polynomial. Conclude $I(C) = (x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2)$.

$I(C) = (x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2)$

(b)

$$0 \to R(-3)^2 \xrightarrow{\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_4 \end{bmatrix}} R(-2)^3 \xrightarrow{\left[\begin{vmatrix} x_2 & x_3 \\ x_3 & x_4 \end{vmatrix}, \quad -\begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix}, \quad \begin{vmatrix} x_1 & x_2 \\ x_2 & x_3 \end{vmatrix}\right]} R$$

(c)

$$\mathrm{HS}_S(t) = \frac{1 - 3t^2 + 2t^3}{(1-t)^4} = \frac{1+2t}{(1-t)^2}$$

$$e_R(S) = (1+2t)|_1 = 3.$$

(d) Yes, C, which is parameterized by

$$\Gamma([s:t]) = [s^3 : s^2t : st^2 : t^3],$$

appears to be a curve of degree 3.

Example. This is the first example in Eisenbud's book "The geometry of syzygies".

It shows that syzygies encode significant information about geometry. Let X be a set of 7 points in projective 3-space $\mathbb{P}^3$ over $\boldsymbol{k}$.

Assume no more than 2 points of X are on any line and no more than 3 points of X are on any plane.

Let $I(X)$ be the set of polynomials in $R = \boldsymbol{k}[x_1, x_2, x_3, x_4]$ that vanish on X and S be the homogeneous coordinate ring of X, in other words, let $S = R/I(X)$.

View S as an R-module.

$X \subseteq \mathbb{P}^3$, 7 points in general position, $S = R/I(X)$, $R = \boldsymbol{k}[x_1, \ldots, x_4]$

Eisenbud says there are exactly two distinct configurations for graded Betti numbers of the homogeneous coordinate ring S:

$$0 \longrightarrow R(-5)^3 \longrightarrow R(-4)^6 \longrightarrow \begin{matrix} R(-2)^3 \\ \oplus \\ R(-3) \end{matrix} \longrightarrow R$$

$$0 \longrightarrow R(-5)^3 \longrightarrow \begin{matrix} R(-3)^2 \\ \oplus \\ R(-4)^6 \end{matrix} \longrightarrow \begin{matrix} R(-2)^3 \\ \oplus \\ R(-3)^3 \end{matrix} \longrightarrow R$$

In the first case the points do not lie on any curve of degree 3. In the second case, the ideal J generated by the three quadratic generators of $I(X)$ is the ideal of the unique curve of “degree 3” which contains X and this curve is irreducible. I think that Eisenbud claims that after a change of variables, the unique curve of "degree 3" is the C from the earlier example.

Example. This example shows that the **form** of the syzygies and not just the graded Betti numbers affects the geometry. Consider three homogeneous polynomials g_1, g_2, g_3 of the same degree in $\boldsymbol{k}[x,y]$. Assume that the only polynomials that divide all three g's are the constants. The morphism

$$\begin{array}{rccc} [g_1 : g_2 : g_3] : & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & [a:b] & \mapsto & [g_1(a,b) : g_2(a,b) : g_3(a,b)] \end{array}$$

defines a rational curve C in the projective plane. Information about the singularities of C may be read from the syzygies of $[g_1, g_2, g_3]$. Let φ denote the syzygy matrix; as in the other examples, the g's are the signed maximal order minors of φ. .

The configuration of all singularities that can appear on, or infinitely near, a rational plane quartic are completely determined by two classical formulas:

$$g = \binom{d-1}{2} - \sum_q \binom{m_q}{2} \qquad \text{and} \qquad \sum_{q'} m_{q'} \leq m_p,$$

The formula on the left was known by Max Noether. It gives the genus g of the irreducible plane curve C of degree d, where q varies over all singularities on, and infinitely near, C, and m_q is the multiplicity at q. The formula on the right holds for any point p on any curve C; the points q' vary over all of the points in the first neighborhood of p. The above formulas permit 9 possible singularities on a rational plane curve of degree 4:

Classical name	modern name	multiplicity sequence
Node	A_1	$(2:1,1)$
Cusp	A_2	$(2:1)$
Tacnode	A_3	$(2:2:1,1)$
Ramphoid Cusp	A_4	$(2:2:1)$
Oscnode	A_5	$(2:2:2:1,1)$
A_6-Cusp	A_6	$(2:2:2:1)$
Ordinary Triple Point	D_4	$(3:1,1,1)$
Tacnode Cusp	D_5	$(3:1,1)$
Multiplicity 3 Cusp	E_6	$(3:1)$

The thirteen possible ways to configure the above singularities on a rational plane quartic are given by

$$
\begin{array}{llll}
(3:1,1,1) & (2:2:2:1,1) & (2:2:1,1),(2:1,1) & (2:1,1)^3 \\
(3:1,1) & (2:2:2:1) & (2:2:1,1),(2:1) & (2:1,1)^2,(2:1) \\
(3:1) & & (2:2:1),(2:1,1) & (2:1,1),(2:1)^2 \\
 & & (2:1)^3 & (2:2:1),(2:1).
\end{array}
$$

If the singularity configuration of $\mathcal{C}$ is given by is equal to

$$(2:2:2:1,1) \qquad \text{or} \qquad (2:2:2:1)$$

then

$$\varphi = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_1 \\ 0 & Q_3 \end{bmatrix}.$$

If the singularity configuration of C is given by is equal to

$$(2:2:2:1,1) \qquad \text{or} \qquad (2:2:2:1)$$

then

$$\varphi = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_1 \\ 0 & Q_3 \end{bmatrix}.$$

(a) If the singularity configuration is $(2:2:2:1,1)$, then Q_3 is the product of 2 non-associate linear forms in B. The only singularity on C is $[0:0:1]$ and this singularity is an oscnode.

(b) If the singularity configuration is $= (2:2:2:1)$, then Q_3 is a perfect square. The only singularity on C is $[0:0:1]$ and this singularity is an A_6-cusp.

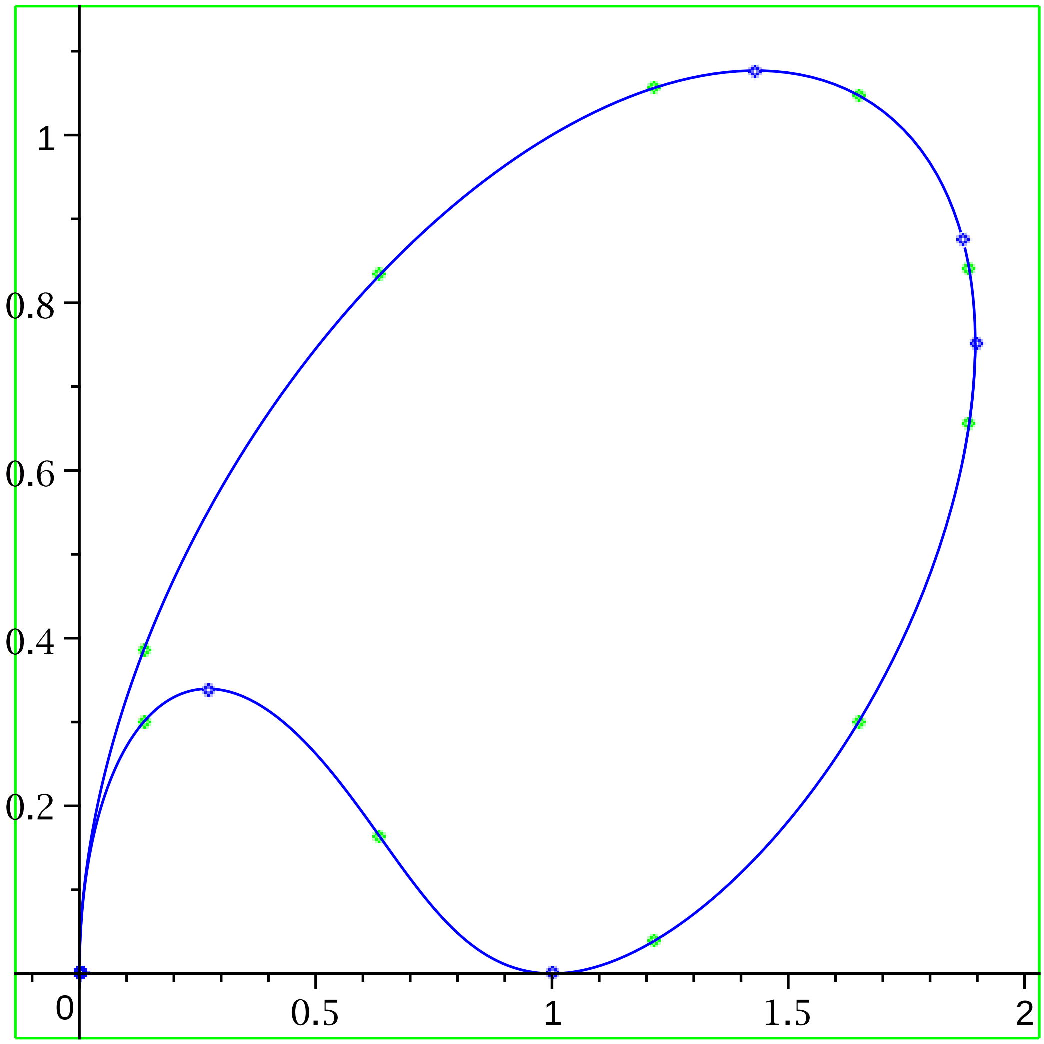
Let $\mathcal{C}$ be parameterized by the m.o.m. of $\varphi = \begin{bmatrix} -(x-y)^2 & -xy \\ x^2 & -(x-y)^2 \\ 0 & x^2 \end{bmatrix}$. This singularity has one branch only after 3 blowups. It is an "A_6-cusp". The picture looks like a "thought caption".

Let $\mathcal{C}$ be parameterized by the m.o.m. of $\varphi = \begin{bmatrix} x^2+xy+y^2 & y^2-x^2 \\ x(x+2y) & x^2+xy+y^2 \\ 0 & x(x+2y) \end{bmatrix}$.
This singularity has two branchs after 3 blowups. It is an "oscnode". The picture looks like a "bowtie".

with(*algcurves*);

[*AbelMap*, *Siegel*, *Weierstrassform*, *algfun_series_sol*, *differentials*, *genus*, *homogeneous*, *homology*, *implicitize*, *integral_basis*, *is_hyperelliptic*, *j_invariant*, *monodromy*, *parametrization*, *periodmatrix*, *plot_knot*, *plot_real_curve*, *puiseux*, *singularities*] **(1)**

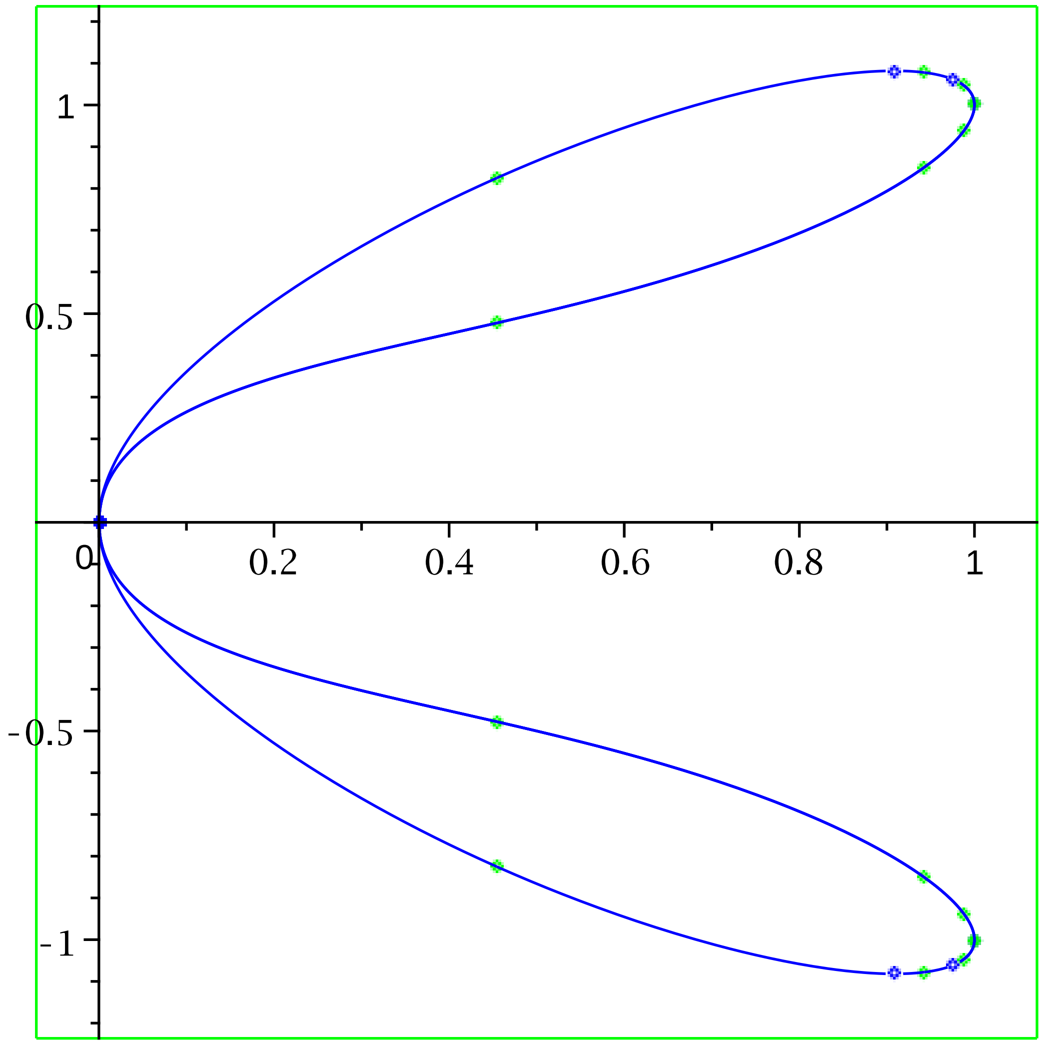
plot_real_curve((x^2 + y^2)^2 - x^3*y - 2*x^3 - 2*x*y^2 + x^2, x, y);



with(algcurves);

[AbelMap, Siegel, Weierstrassform, algfun_series_sol, differentials, genus, homogeneous, homology, implicitize, integral_basis, is_hyperelliptic, j_invariant, monodromy, parametrization, periodmatrix, plot_knot, plot_real_curve, puiseux, singularities] **(1)**

plot_real_curve(x^3*(x-1)+(y^2-x)^2, x, y);



So much for the questions "What is a syzygy?" and "Why do I care?"

Now it is time to answer, "So how does one find syzygies?"

I propose four methods.

1. One can build complicated resolutions from simpler resolutions. If $M = M_1/M_2$ and one knows how that $\mathbb{B}$ resolve M_1 and $\mathbb{T}$ resolves M_2, then M is resolved by the mapping cone of

$$\begin{tikzcd}
\cdots \arrow[r] & T_2 \arrow[r, "t_2"] \arrow[d, "c_1"] & T_1 \arrow[r, "t_1"] \arrow[d, "c_1"] & T_0 \arrow[d, "c_0"] \\
\cdots \arrow[r] & B_2 \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] & B_0 .
\end{tikzcd}$$

2. One can use methods from Algebraic Topology to find syzygies. The starting point for this idea is the following Theorem of Hochster. If I is a square-free monomial ideal in R, then I corresponds to a simplicial complex Δ and the Betti numbers of R/I may be computed from the reduced simplicial homology of Δ.

3. A computer uses Gröbner bases to compute syzygies. One orders the monomials in R. Every polynomial now has a leading term. One picks a generating set

$$g_1, \dots, g_N \tag{2}$$

for I with the property that $\text{leading-term}(g_1), \dots, \text{leading-term}(g_N)$ is also a generating set for the ideal $(\{\text{leading-term}(f) \mid f \in I\})$. The generating set (2) is called a Gröbner basis for I. It is not a minimal generating set for I, but it is wonderful for making calculations.

4. One can use the Geometric method for finding syzygies. Let Y be a subvariety of $\mathbb{P}^{n-1}$. To employ the geometric method for finding the resolution of $R/I(Y)$ by free R-modules, one first resolves the singularity. Let $Z \to Y$ be a desingularization of Y. Now, Z is smooth; so, Z is defined by a regular sequence and the coordinate ring of Z is resolved by the Koszul complex for this regular sequence. The Koszul complex resolution is the most completely understood and straightforward of all possible resolutions. To resolve the homogeneous coordinate ring of Y, one "need only" push the Koszul complex down to Y along the desingularization.