PRE-TALK

Let $\boldsymbol{k}$ be a field. The **talk** is about

$$(*)\quad \begin{cases} \text{standard-graded,} \\ \text{Artinian,} \\ \text{Gorenstein,} \\ \boldsymbol{k}\text{-algebras} \\ \text{of embedding dimension 4, and} \\ \text{socle degree 3.} \end{cases}$$

The purpose of the **pre-talk** is to make sense of those words and to describe a parameterization of (*). It turns out that, in some sense,

the set of things described by (*) is parameterized by the set of cubic surfaces in $\mathbb{P}^3$,

I am amused by the fact that there have already been two talks in this seminar about cubic surfaces in $\mathbb{P}^3$.

• A standard-graded Artinian Gorenstein $\boldsymbol{k}$-algebras of embedding dimension 4 and socle degree 3 is a ring of the form

$$A = \boldsymbol{k}[x,y,z,w]/I$$

where the variables have degree 1, I is homogeneous, $A_4 = 0$, $\dim_{\boldsymbol{k}} A_3 = 1$, $\dim A_1 = 4$, and if a_i is a non-zero element of A_i for $i = 1$ or 2, then $a_i A_{i-3} \neq 0$.

• If you wanted to build such a thing, you might take an element $\theta \in P_3 \setminus I$ (where $P = \boldsymbol{k}[x,y,z,w]$); then for each element in your favorite basis for P_3, you would identify $\alpha_{\text{element}} \in \boldsymbol{k}$ with

$$\text{element} - \alpha_{\text{element}}\theta \in I.$$

At this point, you would know all of I and A.

Example. Suppose $x^3 \notin I$ but $y^3 - x^3, z^3 - x^3, w^3 - x^3, m - 0x^3 \in I$ for all cubic monomials m in x,y,z,w with m not a perfect cube. Observe that

$$(xy, xz, xw, yz, yw, zw, y^3 - x^3, z^3 - x^3, w^3 - x^3) \subseteq I.$$

Notice xy sends every element of P_1 to I; so $xy \in I$. A linear algebra calculation (it is implemented in Macaulay2 as fromDual) shows that we have identified all of I. I will carry this calculation out in a second.

• I want to clean this idea up. The game of "take $\theta \in P_3 \setminus I$ and for each element in your favorite basis for P_3, identify $\alpha \in \boldsymbol{k}$ with element $- \alpha\theta \in I$" is a very complicated way of saying pick a non-zero homomorphism $\phi : P_3 \to \boldsymbol{k}$. The cleanest way to get ϕ (which is a map $P_3 \to \boldsymbol{k}$) to also give information about $P_1 \to \boldsymbol{k}$ is to consider

$$\bigoplus_i \operatorname{Hom}_{\boldsymbol{k}}(P_i, \boldsymbol{k})$$

as a module over P with action: if $u_i \in P_i$ and $w_j \in \operatorname{Hom}_{\boldsymbol{k}}(P_j, \boldsymbol{k})$, then

$$u_i w_j \in \operatorname{Hom}_{\boldsymbol{k}}(P_{j-i}, \boldsymbol{k})$$

and

$$u_i w_j(u_{j-i}) = w_j(u_{j-i} u_i).$$

• In the above example,

$$\phi : P_3 \to \boldsymbol{k}$$

sends x^3, y^3, z^3, w^3 to 1 and all other cubic monomials to zero.

I want to calculate

$$\operatorname{ann}_P(\phi) = \cup_i \{u_i \in P_i \mid u_i \phi = 0\}.$$

Observe that

$$P_i \phi = 0 \text{ for } 4 \le i.$$

We see that

$$\begin{aligned} &\{u_3 \in P_3 \mid u_3 \phi = 0\} = \ker \phi \\ &= (x^3 - y^3, x^3 - z^3, x^3 - w^3, \{u | \text{u is a cubic monomial but not a perfect square}\}). \end{aligned}$$

We see that xy, xz, xw, yz, yw, zw all kill ϕ and $x^2\phi$ (which sends x to 1 and the other variables to 0), $y^2\phi$, $z^2\phi$, and $w^2\phi$ are linearly independent elements of $\operatorname{Hom}_{\boldsymbol{k}}(P_1, \boldsymbol{k})$. So

$$\{u_2 \in P_2 \mid u_2 \phi = 0\} = (xy, xz, xw, yz, yw, zw).$$

Similarly, $x\phi$ (which sends x^2 to 1 and the other quadratic monomials to 0), $y\phi$, $z\phi$, and $w\phi$ are linearly independent elements of $\operatorname{Hom}_{\boldsymbol{k}}(P_2, \boldsymbol{k})$; so,

$$\{u_1 \in P_1 \mid u_1 \phi = 0\} = 0.$$

$$\{u_0 \in P_0 \mid u_0 \phi = 0\} = 0.$$

So

$$\operatorname{ann}_P \phi = (xy, xz, xw, yz, yw, zw, y^3 - x^3, z^3 - x^3, w^3 - x^3).$$

Theorem. (Macaulay 1916) Let $\boldsymbol{k}$ be a field, U be a finite dimensional vector space over $\boldsymbol{k}$, P be the polynomial ring $P = \operatorname{Sym}_\bullet U$, $\mathfrak{m}$ be the maximal homogeneous ideal of P, and D be the P-module $D = \bigoplus_i \operatorname{Hom}_{\boldsymbol{k}}(\operatorname{Sym}_i(U), \boldsymbol{k})$. Then there is a one-to-one correspondence

$$\left\{ I \,\middle|\, \begin{array}{l} I \text{ is a homogeneous } \mathfrak{m}\text{-primary} \\ \text{ideal of } P \end{array} \right\} \leftrightarrow \left\{ M \,\middle|\, \begin{array}{l} M \text{ is a homogeneous} \\ \text{finitely generated } P\text{-submodule of } D \end{array} \right\}$$

with

$$I \longmapsto \operatorname{ann}_D(I) \quad \text{and}$$

$$\operatorname{ann}_P(M) \longleftarrow M .$$

Furthermore,

(a) the correspondence is a duality in the sense that

$$\operatorname{ann}_P(\operatorname{ann}_D(I)) = I \quad \text{and} \quad \operatorname{ann}_D(\operatorname{ann}_P(M)) = M,$$

(b) the ideal I defines a Gorenstein quotient if and only if $\operatorname{ann}_D I$ is cyclic,

(c) if P/I is Gorenstein, then the socle degree of P/I is equal to the generator degree of $\operatorname{ann}_D(I)$.

• M (or a generator of M) is usually called a *Macaulay Inverse System* for $\mathrm{ann}_P M$.

• I usually write D_iU^* in place of $\mathrm{Hom}_{\boldsymbol{k}}(\mathrm{Sym}_i U, \boldsymbol{k})$.

• If $x_1, \dots, x_n$ is a basis for U, then the set of monomials of degree i in $x_1, \dots, x_n$ (I write $\binom{x_1,\dots,x_n}{i}$ for this set of monomials.) is a basis for $\mathrm{Sym}_i U$ and

$$\{m^* \mid m \in \binom{x_1,\dots,x_n}{i}\}$$

is the basis for D_iU^* which is dual to $\binom{x_1,\dots,x_n}{i}$.

• Observe that

$$x_i(m^*) = \begin{cases} (m/x_i)^* & \text{if } x_i | m \\ 0 & \text{otherwise.} \end{cases}$$

• In my talk, there is no need to have a product structure in $D_\bullet(U^*)$. But notice that if one puts a multiplication on $D_\bullet(U^*)$, then one must make sense of $x(x^*x^*)$. Surely, the action should be the product rule; so

$$x(x^*x^*) = 2x^*$$
$$x((x^2)^*) = x^*$$

We decide that the object previously written as $(x^2)^*$ should be equal to $(1/2)x^*x^*$. The moral is that $D = D_\bullet(U^*)$ should be given the structure of a divided power algebra. That is, for each homogeneous element w of D (of positive degree), a sequence of elements $\{w^{(n)}\}$ should exist in D. The rules that these elements $w^{(n)}$ satisfy are the same as the rules that $\{(1/n!)w^n\}$ satisfies whenever $(1/n!)w^n$ is defined. See Gulliksen and Levin (Homology of local rings) or Eisenbud (View) for the list of rules satisfied by a Divided Power Algebra. For example $(w_1 + w_2)^{(2)}$ should equal

$$(1/2)(w_1 + w_2)^2 = (1/2)(w_1^2 + 2w_1w_2 + w_2^2),$$

and this should equal $w_1^{(2)} + w_1w_2 + w_2^{(2)}$. So one of the axioms of Divided Power Algebra is

$$(w_1 + w_2)^{(2)} = w_1^{(2)} + w_1w_2 + w_2^{(2)}.$$

• An object of (*) is equal to $\mathrm{ann}_P \phi$ for some $\phi \in D_3U^*$ where $\dim_{\boldsymbol{k}} U = 4$. One can think of $D_\bullet U^*$ as a "divided power ring" in four variables. In characteristic zero, a divided power ring "is" a polynomial ring. In this "sense" an element in D_3U^* is a 3-form in four variables; hence a "cubic surface".

• If U is a vector space, then $\bigwedge^{2\bullet} U$ is another algebra with a divided power structure. In particular, if we follow the above formula, we obtain

$$(x_{12}e_1 \wedge e_2 + x_{13}e_1 \wedge e_3 + x_{14}e_1 \wedge e_4 + x_{23}e_2 \wedge e_3 + x_{24}e_2 \wedge e_4 + x_{34}e_3 \wedge e_4)^{(2)}$$
$$= \mathrm{Pf}\begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix}(e_1 \wedge e_2 \wedge e_3 \wedge e_4).$$