GEORGIA SOUTHERN TALK 2018

Throughout the talk R is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\boldsymbol{k} = R/\mathfrak{m}$. The plan of the talk is to investigate $\boldsymbol{k}$ as an R-module in order to deduce properties of R.

The ring R is Noetherian means that every ideal of R is finitely generated.

The ring R is local means that $\mathfrak{m}$ is the only maximal ideal of R.

A good example of a c.n.l.r. to keep in mind is: Let V be a variety in $\mathbb{A}^n$, p be a point on V, $\mathcal{R} = \boldsymbol{k}[x_1,\dots,x_n]/I(V)$ be the coordinate ring of V, P be the ideal of $\mathcal{R}$ which consists of all elements of $\mathcal{R}$ which vanish at p and $R = \mathcal{R}_P = (\mathcal{R}\setminus P)^{-1}\mathcal{R}$ be the ring of rational functions on V which are defined at p.

A second example of a c.n.l.r. is $R = \boldsymbol{k}[x_1,\dots,x_n]/I$, where I is primary to $(x_1,\dots,x_n)$.

The first time the R-module structure of $\boldsymbol{k}$ was used to give important information about R occurred in the 1950's. It has always been known what it means to be a smooth point on algebraic variety. (It means that the Jacobian matrix at the point has the right rank.) In the 1930's Krull recognized that rings with the property that

$$\mu(\mathfrak{m}) = \dim R \tag{1}$$

were pretty special. (One has $\dim R \le \mu(\mathfrak{m})$ by the Krull principal ideal theorem. Equality holds in an extremal situation. Now-a-days when equality holds, we say that R is a regular local ring. I do not know what name, if any Krull gave the ring.) In the 1940's Zariski recognized that rings with (1) roughly corresponded to smooth points (at least if $\boldsymbol{k}$ is a perfect field). In the 1950's there was a very strong movement that properties should be defined in such a way that they can be checked locally. It is hard to see that the property (1) localizes. There is a direct argument that shows that if R satisfies (1), then localizations of R also satisfy (1). This argument is due to Nagata; it is long and hard, and it came after the observation of Serre that I am about to describe. In the 1950's, Serre proved that R satisfies (1) if and only if the global dimension of R is finite. It is quite clear that the hypothesis R has finite global dimension localizes. Serre's proof is straightforward (if one knows homological algebra). If R satisfies (1), then the Koszul complex is a finite resolution of $\boldsymbol{k}$ by free R-modules. Use the fact that $\mathrm{Tor}_R(-,-)$ can be computed using either component to see that $\mathrm{pd}_R(-) < \infty$. Serre's proof was the first time homological algebra had been used to prove an important new result in commutative algebra/ algebraic geometry. Suddenly mainstream commutative algebraists and algebraic geometers started to take Cartan-Eilenberg and homological algebra seriously. The Koszul complex was borrowed directly from Algebraic Topology. In Algebraic Topology, the Koszul complex is the simplicial chain complex associated to a simplex.

Rather than write a Koszul complex on the board (This is a project that amuses me always; but I am not sure it conveys much information to the audience), I will just write the crude information: the ranks of the Betti numbers. The Koszul complex is an exterior algebra; so the Betti numbers are binomial coefficients

$$0 \to R^{\binom{n}{n}} \to \cdots \to R^{\binom{n}{3}} \to R^{\binom{n}{2}} \to R^n \to R \to 0$$

If I am going to talk about (usually infinite) sequences of Betti numbers; I should organize these sequences. A combinatorist would naturally use the generating function:

$$P^R(t) = \sum_{i=0}^{\infty} \beta_i t^i$$

where the minimal resolution of $\boldsymbol{k}$ by free R-modules is

$$\cdots \to R^{\beta_1} \to R^{\beta_0} \to 0.$$

So Serre's Theorem says

Theorem. The local ring R of dimension n is regular if and only if $P^R(t) = (1+t)^n$.

The next theorem along these lines is due to Tate and then Gulliksen

Theorem. The local ring R is a complete intersection if and only if $P^R(t)$ has the form $\frac{(1+t)^n}{(1-t^2)^m}$ for some integers m and n.

Tate's contribution (in the inaugural issue of IJM) is to say, "One can always get a DG algebra which is a resolution of $\boldsymbol{k}$ by free R-modules." Start with

$$R \to \boldsymbol{k} \to 0$$

Adjoin (exterior) variables of degree 1 to kill the first homology:

$$R\langle e_1, \ldots, e_n \rangle \to \boldsymbol{k} \to 0$$

This much is just the Koszul complex.
Adjoin (divided power variables) of degree 2 to kill the second homology:

$$R\langle e_1, \ldots, e_n, T_1, \ldots, T_m \rangle \to \boldsymbol{k} \to 0$$

Adjoin (exterior variables) of degree 3 to kill the third homology:

$$R\langle e_1, \ldots, e_n, T_1, \ldots, T_m, U_1, \ldots, U_\ell \rangle \to \boldsymbol{k} \to 0$$

etc.

Tate proved that if R is a complete intersection, then the Tate resolution is

$$R\langle e_1, \ldots, e_n, T_1, \ldots, T_m \rangle \to \boldsymbol{k} \to 0.$$

Gulliksen proved that the Tate complex is always a minimal resolution. He also proved that the 2-step Tate complex is a resolution if and only if R is a complete intersection. At this point the combinatorial calculation is easy:

$$R\langle e_1, \ldots, e_n; T_1, \ldots, T_m \rangle = R\langle e_1, \ldots, e_n \rangle \otimes R\langle T_1, \ldots, T_m \rangle.$$

The Hilbert series for $R\langle e_1, \ldots, e_n \rangle$ is $(1+t)^n$ and the Hilbert series for $R\langle T_1, \ldots, T_m \rangle$ is $\frac{1}{(1-t^2)^m}$.

Still back in the 1950's Serre asked (in Local Algebras and Multiplicities LNM 11) and Kaplansky asked (apparently in class) if $P^R(t)$ is always a rational function.

Evidence and Comments:

- All known examples were rational functions.

• Serre had produced a coefficient-wise upper bound which is a rational function:

$$P^R(t) \preceq \frac{(1+t)^{\operatorname{edim}(R)}}{1-\sum\limits_{1\le j}\dim_{\boldsymbol{k}} \mathrm{H}_j(K^R)t^{j+1}}.$$

• It is always wonderful to describe something infinite with finite data.

• Structural information can be read from the denominators if the Poincaré series is rational. How fast do the Betti numbers grow? Are they always eventually non-decreasing?

Serious effort was put into proving that all Poincaré series are rational in Paris, Moscow, Stockholm, Oslo, Chicago.

Most of the results were reduction results "It suffices to show ..."

Here are some particularly noteworthy approaches.

• Jack Shamash (PhD Chicago 1966) wrote a series of 4 papers "The Poincaré series of a local ring II, III, IV" that describe change of ring formulas for Poincaré series. These papers contain many of the ideas that became "Eisenbud operators" and "matrix factorization".

• Golod (Moscow) characterized the rings whose Poincaré series attain the Serre upper bound in terms of Massey operations. Now-a-days these rings are called Golod rings and they continue to be an object of study.

• Eagon (Chicago PhD) found a resolution of $\boldsymbol{k}$ by free R modules, for all R. His resolution has the Betti numbers of the Serre upper bound; but his resolution is usually not minimal. It is minimal precisely if R is a Golod ring. (Eagon's resolution provides a constructive proof of Serre's upper bound Theorem.)

• Gulliksen (Oslo) and Levin (Chicago PhD) wrote the wonderful Queen's Lecture notes "Homology of local rings" (1969). This is the only "published" version of of Eagon's resolution.

• Roos (Stockholm) connected the study of $P^R(t)$, when R is Artinian to the study of $\sum_i \dim_{\boldsymbol{k}} \mathrm{H}_i(\Omega X)t^i$ where X is a finite simply-connected CW complex with cohomology ring R and $\Omega(-)$ means loop space.

Then one day in 1979 or 1980 Richard Stanley went to tea at MIT and he asked one of the graduate students, David Anick, "David, what are you doing these days?" Anick was taking a course in Algebraic Topology with Munkres and in answer to a homework question he had come across a really weird CW-complex, the generating function of the homology of its loop space was not a rational function. This problem sounded familiar to Stanley; he promised to write to his friends in Stockholm. Anick flew to Stockholm. His homework solution/PhD thesis is published in the Stockholm preprint series; his work appears in Comptes Rendus; the paper is published in the Annals; etc.

In the mean time, people simplified the Anick example (although it wasn't very complicated) and gave examples with smaller parameters.

- In 1983, Bøgvad gave a Gorenstein ring with an irrational Poincaré series.
- In 2015 Löfwall-Lundqvist-Roos give a Gorenstein numerical semigroup ring

  $$\boldsymbol{k}[\{t^i | i \in \{36,48,50,52,56,60,66,67,107,121,129,135\}\}]$$

  with irrational Poincaré series.
- In 2017 Roos gave a ring with embedding dimension 4 with an irrational Poincaré.

On the other side,

- In 1980, Judy Sally proved stretched rings have rational Poincaré series
- In 1988, Avramov-Kustin-Miller proved that rings with small embedding codimension or small linking number have rational Poincaré series.
- In 2009 Elias-Valla proved that almost stretched rings have rational Poincaré series
- In 2009 Casnati-Notari rings with multiplicity at most 10 have rational Poincaré series

Nonetheless, in 2012 (at the introductory workshop for the special year in Commutative Algebra at MSRI – Mathematical Sciences Research Institute) Irena Peeva observed Poincaré series have been studied for fifty years. It has been known for thirty years that some Poincaré series are irrational; yet, "we still do not have a feel for which of the following cases holds.

(a) Most Poincaré series are rational, and irrational Poincaré series occur rarely in specially crafted examples.
(b) Most Poincaré series are irrational, and there are some nice classes of rings (she said Golod and complete intersection, I would add small embedding codepth) where we have rationality.
(c) Both rational and irrational Poincaré series occur widely.

One would like to have results showing whether the Poincaré series are rational generically, or are irrational generically."

In 2014, Marilena Rossi and Liana Şega took up Peeva's challenge and proved that generic Artinian Gorenstein rings have rational Poincaré series.

Şega, Vraciu, and I extended the Rossi-Şega result. Artinian Gorenstein rings have socle dimension one. We allow socles of arbitrary dimension but we put other restrictions on the socle.

Theorem. (2018, Kustin, Şega, Vraciu) Let R be a generic Artinian ring with top socle degree s. If s is odd, $5 \le s$, and $\underline{\operatorname{socle} R \cap \mathfrak{m}^{s-1} = \mathfrak{m}^s}$, then R has rational Poincaré series.

(The socle of the local ring $(R,\mathfrak{m})$ is $0 :_R \mathfrak{m} = \{r \in R \mid \mathfrak{m}r = 0\}$. "Socle" is a Greek word which means "base".)

Our proof does not do anything with even top socle degree; indeed, the main step in our proof fails when the top socle degree is even. Bøgvad's example has socle degree three. Şega-Rossi had to give a special argument for socle degree 3; the special information they used is not available to us. I don't know what to make of the funny underlined hypothesis.

The main technique of proof is to exhibit a Golod homomorphism from a complete intersection onto R. (Golod homomorphisms were defined by Levin and developed by Avramov. This is a relative form of the concept of Golod rings. The idea that morphisms have properties (rather than just objects have properties) is due largely to Grothendieck.) We compute in the algebra

$$\mathrm{H}_\bullet(K^R) = \operatorname{Tor}_\bullet^P(R,\boldsymbol{k}) = F \otimes_P \boldsymbol{k},$$

where $R = P/I$, P is regular local, and F is a resolution of R by free P-modules. We prove that there is an element $g \in \operatorname{Tor}_1$ so that

$$g \cdot \operatorname{Tor}_{\mathrm{top}-1} = \operatorname{Tor}_{\mathrm{top}}.$$

If $\mathfrak{m}$ has 4 minimal generators and socle(R) is isomorphic to $\boldsymbol{k}(-4)^2$, then the Betti table for the minimal homogeneous resolution of R by free Q-modules is

	0	1	2	3	4
total :	1	12	19	10	2
0 :	1	.	.	.	.
1 :	.	.	.	.	.
2 :	.	12	15	.	.
3 :	.	.	4	10	.
4 :	.	.	.	.	2

It is clear that $\operatorname{Tor}_1 \cdot \operatorname{Tor}_3 = 0$, but $\operatorname{Tor}_4 \neq 0$.

What does generic mean? Well there are two ways to parameterize Artinian algebras.

- Fix the variables. Fix the degrees of the generators. The coefficients then live in a giant affine space. …
- Each Artinian algebra is defined by the system of differential operators that annihilates it. Fix a basis of differential operators. The coefficients of the differential operators live in a giant affine space. …

The advantage of the differential operator approach is that on a dense open set of the parameter space, the corresponding algebras are compressed; that is, they have the maximum possible length for the given socle degrees. So actually, the theorems are established for "compressed rings" rather than "generic rings", except for the Rossi-Şega result when the socle degree is 3.

- An Example of higher Massey product: Suppose z_1, z_2, z_3 are cycles. If all products in homology are zero, then there exist $w_{1,2}$ and $w_{2,3}$ such that $d(w_{1,2}) = z_1z_2$ and $d(w_{2,3}) = z_2z_3$. Now notice that

$$d(w_{1,2}z_3) = d(w_{1,2})z_3 \pm w_{1,2}d(z_3) = z_1z_2z_3 \quad \text{and} \quad d(z_1w_{2,3}) = d(z_1)w_{2,3} \pm z_1d(w_{2,3}) = \pm z_1z_2z_3.$$

  Thus, $z_1w_{2,3} \pm w_{1,2}z_3$ is a cycle which represents the Massey triple product of $[z_1]$, $[z_2]$, and $[z_3]$. It is reasonable to ask if this triple product is zero in homology; that is if the cycle $z_1w_{2,3} \pm w_{1,2}z_3$ is also a boundary.
- A quick explanation of loop space. If (X, x_0) is a pointed topological space, then the loop space of X, $\Omega(X)$ is the set of loops in X which start and stop at x_0. The loop space $\Omega(X)$ is a topological space and also a group.
- A field is perfect if every irreducible polynomial over the field has distinct roots in the algebraic closure.