Graduate Colloquium Talk about Math 746. Spring 2018

Math 746 is about commutative Noetherian rings. (Noetherian means that every ideal is finitely generated.)

The basic examples are the ring of integers or a field. If R is a c.n.r, then $R[x_1, \dots, x_n]$, R/I, and $S^{-1}R$ all are c.n.r. If I is an ideal in a c.n.r R, then one can use I to put a topology on R. Two elements of R are close together if their difference is in a high power of I. One can then complete R in this "I-adic" topology; thereby creating p-adic integers and formal power series rings. These rings are also c.n.r.

A central invariant of a c.n.r is its Krull dimension. The Krull dimension of the c.n.r R is the length of the longest chain of prime ideals in R. A field has Krull dimension zero: (0). The ring of integers has Krull dimension one: $(0) \subseteq (2)$, the polynomial ring $k[x, y]$ has Krull dimension two: $(0) \subseteq (x) \subseteq (x, y)$. (In each case, I have identified a chain of prime ideals of maximum length.)

I give a hint about how important Krull dimension is.

One of my favorite rings is $R = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. (Maybe you should insist that $(f_1, \dots, f_r)$ is a prime ideal.) The common zero set of $f_1 = \cdots = f_r = 0$ in n-space is a geometric object $X = V(f_1, \dots f_r)$ and R is the coordinate ring of X (the ring of polynomial functions from X to k). It turns out that the Krull dimension of R equals the dimension of X as a manifold.

One way to get my fingers on $\dim X$ is to say: a finite set of points has dimension 0. If $\dim X \neq 0$ and $X \cap V(g_1)$ is a finite set of points for some g_1, then X has dim =1. If $2 \leq \dim X$ and $X \cap V(g_1) \cap V(g_2)$ is a finite set of points, for some g_1, g_2 then $\dim X = 2$. There is an approach to Krull dimension that mimics this procedure.

There is a combinatorial interpretation of Krull dimension. Usually R is too big to count. So I find something I can count. Let $\mathfrak{m}$ be a maximal ideal of R. It turns out that $R/\mathfrak{m}^s$ has a finite composition series for each s and the length of this composition series is denoted $\ell(R/\mathfrak{m}^s)$ and is called the length of $R/\mathfrak{m}^s$. (In many cases ℓ is a vector space dimension.) It turns out that $\ell(R/\mathfrak{m}^s)$ is a polynomial in s for all large s and the degree of this polynomial is also the Krull dimension of R.

I will teach from Eisenbud's Commutative Algebra, with a view toward Algebraic Geometry. It is a very chatty book with lots of excercises, lots of motivation for each topic, lots of history. It is far removed from books that consist only of Theorems and proofs.