CHAPTER 2 EXERCISE SOLUTIONS FROM DYER-EDMUNDS "REAL TO COMPLEX ANALYSIS"

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1. PROBLEM 2.1.1

Set \( f(t) := \frac{\nu}{p} + \frac{1}{p} - 1 \). We see that \( f'(t) = t^{\nu-1} - 1 \) has a real root at \( t = 1 \) and is positive for \( t > 1 \), so that \( t \) is a minimum. Plugging in \( t = 1 \),

\[
f(1) = 0
\]
So that we may take \( t = \frac{a}{b^{\frac{1}{p-1}}} \) to see:

\[
f\left( \frac{a}{b^{\frac{1}{p-1}}} \right) = \frac{a^p}{pb^\frac{p}{p-1}} + \frac{1}{p'} - \frac{a}{b^{\frac{1}{p-1}}} \geq 0
\]

\[
\implies \frac{a^p}{p} + \frac{b^{p'}}{p'} \geq ab
\]

With equality if and only if \( a = b^{\frac{1}{p-1}} \). Now, without loss of generality we may assume (by homogeneity)

\[
\left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} = \left( \sum_{k=1}^{n} |y_k|^{p'} \right)^{1/p'} = 1
\]

Using Young’s inequality (as proved above):

\[
\sum_{k=1}^{n} |x_k y_k| \leq \sum_{k=1}^{n} \left( \frac{|x_k|^p}{p} + \frac{|y_k|^{p'}}{p'} \right)
\]

\[
= \frac{1}{p} \sum_{k=1}^{n} |x_k|^p + \frac{1}{p'} \sum_{k=1}^{n} |y_k|^{p'}
\]

\[
= \frac{1}{p} + \frac{1}{p'} = 1
\]

Yielding Hölder’s inequality. Now, to get Minkowski’s, we see:

\[
\sum_{k=1}^{n} |x_k + y_k|^p \leq \sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1}
\]

\[
\leq \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \left( \sum_{k=1}^{n} |x_k + y_k|^{p-1} \right)^{1-1/p}
\]

\[
+ \left( \sum_{k=1}^{n} |y_k|^p \right)^{1/p} \left( \sum_{k=1}^{n} |x_k + y_k|^{p-1} \right)^{1-1/p}
\]

\[
\implies \left( \sum_{k=1}^{n} |x_k + y_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left( \sum_{k=1}^{n} |y_k|^p \right)^{1/p}
\]

This of course then yields that \( d_p \) is a metric since the only nontrivial fact is the satisfaction of the triangle inequality.
2. **Problem 2.1.2**

Note of course by Hölder’s inequality,

\[ ||fg||_r \leq ||f||_p ||g||_q \]

When \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). In particular,

\[
\left\| f + g \right\|_p^p = \int_a^b |f + g|^p \, dx \\
\leq \int_a^b |f||f + g|^{p-1} \, dx + \int_a^b |g||f + g|^{p-1} \, dx \\
\leq \left( ||f||_p ||f + g||_p^{p-1} + ||g||_p ||f + g||_p^{p-1} \right) \\
\Rightarrow \left\| f + g \right\|_p \leq \left( ||f||_p + ||g||_p \right)
\]

Symmetry is obvious. It remains to prove nondegeneracy. Suppose \( d(f, g) = 0 \). By continuity we immediately see that \( f \equiv g \), since if not, we may find a neighborhood where the difference \( f - g \) is strictly positive, yielding that \( d(f, g) > 0 \). Hence, this is a norm.

3. **Problem 2.1.3**

Merely note that \( \ell^p \) is precisely \( L^p(\# , \mathbb{N}) \) where \( \# \) is the counting measure on the integers. By Minkowki’s inequality, the metric properties are trivial.

4. **Problem 2.1.4**

Note first that \( d(x, y) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \), so this is well defined and finite. If \( d(x, y) = 0 \), then \( x_n = y_n \) for every \( n \), since else the above sum would be strictly positive at some point.
To get the triangle inequality, note that the map \( t \mapsto \frac{t}{t+1} \) is increasing. We have the triangle inequality for the standard norm, so

\[
| x_n - z_n | \leq | x_n - y_n | + | y_n + z_n |
\]

\[
\Rightarrow \frac{| x_n - z_n |}{1 + | x_n - z_n |} \leq \frac{| x_n - y_n |}{1 + | x_n - y_n |} + \frac{| y_n - z_n |}{1 + | y_n - z_n |}
\]

Which, when applied to each term in the sum, yields

\[
d(x, z) \leq d(x, y) + d(y, z)
\]

As desired. Again, symmetry is obvious.

5. Problem 2.1.5

We see that

\[
a - c = k_1 p^{t(a,c)}
\]

and

\[
a - c = a - b + b - c
\]

\[
= k_2 p^{t(a,b)} + k_3 p^{t(b,c)}
\]

We may suppose that \( t(b, c) \leq t(a, b) \), implying

\[
k_1 p^{t(a,c)} = p^{t(b,c)}(k_2 p^{t(a,b)} - t(b,c) + k_3)
\]

So that \( t(a, c) \geq t(b, c) \). By symmetry we deduce \( t(a, c) \geq \max\{t(a, b), t(b, c)\} \).

Then, certainly

\[
\frac{1}{p^{t(a,c)}} \leq \frac{1}{p^{t(b,c)}} < \frac{1}{p^{t(a,b)}} + \frac{1}{p^{t(b,c)}}
\]

So that \( d(a, c) \leq d(a, b) + d(b, c) \). The other two properties are tautological.

6. Problem 2.1.6

(i). \( \mathbb{N} \) is closed.

(ii). \( N := \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \) is neither as \( \overline{N} = \mathbb{N} \cup \{0\} \).
(iii). $\mathbb{Q}$ is neither open nor closed, since it is dense in the reals and contains no neighborhood about its points by density of irrationals.

(iv). Set $N := \left\{ (-1)^n \left( 1 + \frac{1}{n} \right) \mid n \in \mathbb{N} \right\}$. This is neither closed nor open as $\overline{N} = N \cup \{0\}$ and clearly it contains no neighborhood about its points.

7. Problem 2.1.7

(i). Let $f(x, y) := x^2 + y^2$. $f$ is continuous, so that $f^{-1}(-\infty, 1)$ is open, and,

$$f^{-1}(-\infty, 1) \cap ((0, \infty) \times \mathbb{R}) \cap (\mathbb{R} \times (0, \infty))$$

is the intersection of finitely many open sets, hence open. But this is precisely the desired set.

(ii). Define $f(x, y) := x + y$. The set of interest is then

$$f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$$

By continuity, this is open.

(iii). Define $f(x, y) := xy$. This is continuous, and the set of interest is

$$f^{-1}(-\infty, 1) \cup f^{-1}(1, \infty)$$

Which is open.

8. Problem 2.1.8

Let $A \subset X$, $X$ discrete. If $x \in A$, then consider $B(x, 1/2)$. Obviously $B(x, 1/2) \subset A$ since only $x \in B(x, 1/2)$. An identical argument then shows that for any $y \in A^c$, $B(y, 1/2) \subset A^c$, so that $A^c$ is open as well. Hence, every set is open and closed.
9. Problem 2.1.9

Assume without loss of generality that \( d(x, z) \geq d(y, z) \). Then, \( k(y, z) \geq k(x, z) \) and by definition \( x_n = z_n \) for \( n < k(x, z) \) and \( y_n = z_n \) for \( n < K(y, z) \). This immediately gives that \( x_n = y_n \) for \( n < \min\{k(x, z), k(y, z)\} \). Inverting,

\[
d(x, y) \leq \max\{d(x, z), d(y, z)\}
\]

\[
\implies d(x, y) \leq d(x, z) + d(y, z)
\]

And the other properties are again tautological.

10. Problem 2.1.10

Assume without loss of generality that \( d(x, z) > d(y, z) \), so that \( d(x, y) \leq d(x, z) \). But,

\[
d(x, z) \leq \max\{d(x, y), d(y, z)\}
\]

and since \( d(x, z) > d(y, z) \), we must have that

\[
d(x, z) \leq d(x, y)
\]

Implying \( d(x, y) = d(x, z) \), and

\[
d(x, y) = \max\{d(x, z), d(y, z)\}
\]

Now suppose that \( y \in B(x, r) \), so that \( d(x, y) < r \). If \( z \in B(y, r) \), then

\[
d(x, z) \leq \max\{d(x, y), d(z, y)\}
\]

\[
< r
\]

So that \( B(y, r) \subset B(x, r) \). By symmetry, we deduce that \( B(x, r) \subset B(y, r) \) as well, yielding equality. Suppose now that \( B(x, r_1) \cap B(y, r_2) \neq \emptyset \), and without loss of generality assume that \( r_1 > r_2 \). Choose \( z \in \emptyset \).
$B(x, r_1) \cap B(y, r_2)$. We see:

$$d(x, y) \leq \max \{d(x, z), d(y, z)\}$$

$$< \max \{r_1, r_2\} = r_1$$

So that $y \in B(x, r_1)$. Then, one immediately sees that for any $z \in B(y, r_1)$,

$$d(x, z) \leq \max \{d(x, y), d(y, z)\}$$

$$< \max \{r_1, r_2\} = r_1$$

Which gives that $B(y, r_2) \subset B(x, r_1)$. Now, given any $x \in X$, the above shows that if $y \notin B(x, r)$, we can find $r_2$ such that $B(x, r) \cap B(y, r_2) = \emptyset$ (since else, we can use the above argument to prove $y \in B(x, r)$). But this immediately gives that $B(x, r)$ is closed, since its complement contains a neighborhood around all of its points. Now, set

$$K := \{y \in X \mid d(x, y) \leq r\}$$

Given $z \in B(y, r)$, we see that

$$d(x, z) \leq \max \{d(x, y), d(y, z)\}$$

$$\leq r$$

So that $B(y, r) \subset K$, and $K$ contains a neighborhood around all of its points, whence $K$ is open.

11. **Problem 2.1.11**

It remains only to show symmetry. Now that

$$d(x, y) \leq d(x, x) + d(y, x)$$

$$= d(y, x)$$

$$d(y, x) \leq d(x, x) + d(y, x)$$

$$= d(x, y)$$

So that $d(x, y) = d(y, x)$. 
12. Problem 2.1.12

We only need show that basis elements of the topology contain each other. For all \( r > 0 \), suppose \( y \in B_1(x, r) \). Then,
\[
    d_1(x, y) < r \implies d_2(x, y) < \beta r
\]
\[
    \implies B_1(x, r) \subset B_2(x, \beta r)
\]
Similarly, if \( y \in B_2(x, r) \),
\[
    d_2(x, y) < r \implies d_1(x, y) < \frac{r}{\alpha}
\]
\[
    \implies B_2(x, r) \subset B_1(x, r/\alpha)
\]
All open sets are hence equal, implying that \( d_1 \) and \( d_2 \) generate the same topology.

13. Problem 2.1.13

Note that \( A^o \cap B^o \) is an open set contained in the intersection \( A \cap B \), so that by definition \( A^o \cap B^o \subset (A \cap B)^o \). For the reverse inclusion,
\[
    (A \cap B)^o \subset A^o, \quad (A \cap B)^o \subset B^o
\]
\[
    \implies (A \cap B)^o \subset A^o \cap B^o
\]
\[
    \implies (A \cap B)^o = A^o \cap B^o
\]
Similarly, We have that \( \overline{A \cup B} \supset \overline{A \cup B} \) by definition. For the reverse inclusion,
\[
    \overline{A} \subset \overline{A \cup B}, \quad \overline{B} \subset \overline{A \cup B}
\]
\[
    \implies \overline{A \cup B} \subset \overline{A \cup B}
\]
\[
    \implies \overline{A \cup B} = \overline{A \cup B}
\]
Now, let \( A = \mathbb{Q} \), \( B = \mathbb{Q}^c \). Then,
\[
    A^o \cup B^o = \emptyset \neq \mathbb{R} = (A \cup B)^o
\]
\[
    \overline{A \cap B} = \emptyset \neq \overline{A \cap B} = \overline{A \cap B}
\]
Now, letting \( f(x, y, z) := \cosh(x + yz) \), we see that \( f \) is clearly continuous. Then, \( D = f^{-1}([2, \infty)) \) is closed by continuity, so that \( D = \overline{D} \).

Now, taking interiors,
\[
D^o = \left( f^{-1}([2, \infty)) \right)^o = f^{-1}(2, \infty) = \{(x, y, z) \mid \cosh(x + yz) > 2\}
\]

14. **Problem 2.1.14**

(i). Closure: \( \{(x, y) \mid 0 \leq x \leq y \leq 1\} \)

Interior: \( \{(x, y) \mid 0 < x < y < 1\} \)

(ii). Closure: \( \{(x, 0) \mid 0 \leq x \leq 1\} \)

Interior: \( \emptyset \)

(iii). Closure: \( \mathbb{R}^2 \)

Interior: \( \emptyset \)

15. **Problem 2.1.15**

Let \( y \in [0, 1] \setminus S \). We need not consider decimal expansions with an infinite string of 9’s at the end, since these have an equivalent finite expansion. Then,

\[
y = 0.b_1 \ldots b_n \ldots, \quad b_1, \ldots, b_{n-1} \in \{0, 1\}
\]

and \( b_n \notin \{0, 1\} \). Then for all \( x \in S \),
\[
|x - y| = \sum_{k=1}^{\infty} \frac{|b_k - a_k|}{10^k} \geq \frac{1}{10^k}
\]

Since at least \( |b_n - a_n| \geq 1 \). But then we see that \( S^c \) is open, so that \( S \) is closed, yielding \( \overline{S} = S \).
16. **Problem 2.1.16**

On a discrete space, $B(x, 1) = B(x, 1)$ since all sets are clopen. However,

$$\{ y \mid d(x, y) \leq 1 \} = X \neq B(x, 1)$$

Whence the result.

17. **Problem 2.1.17**

(i). This is trivial by definition of the subspace topology.

(ii). Note

$$\text{Cl}_Y(S) = \bigcap_{S \subseteq B \text{ closed}} B = \bigcap_{S \subseteq A \text{ closed}} Y \cap A = Y \cap \left( \bigcap_{S \subseteq A \text{ closed}} A \right) = Y \cap \text{Cl}_X(S) = \text{Cl}_X(S)$$

(17.1)

Since $Y$ is closed.

18. **Problem 2.1.18**

Set $y = \lambda x$ for $\lambda \in \mathbb{R}$. Then,

$$f(x, \lambda x) = \begin{cases} \frac{\lambda^2}{1 + \lambda^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 1 \end{cases}$$

Then for any $\lambda \neq 0$,

$$\lim_{x \to 0} f(x, \lambda x) \neq 0$$

So that $f$ is not continuous at 0.
19. **Problem 2.1.19**

Let $\epsilon > 0$. We can find $\delta$ such that $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \frac{1}{\sqrt{2}}$$
$$|g(x) - g(y)| < \frac{1}{\sqrt{2}}$$

Then,

$$||(f(x), g(x)) - (f(y), g(y))||_2 = \left( |f(x) - f(y)|^2 + |g(x) - g(y)|^2 \right)^{1/2}$$
$$< \left( \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} \right)^{1/2}$$
$$= \epsilon$$

So we have continuity.

20. **Problem 2.1.20**

At every point $(x, y) \neq (0, 0)$ we clearly have continuity. Now, to look at $(0, 0)$, set $y = \lambda x$ for $\lambda \in \mathbb{R}$. Then,

$$f(x, y) = \begin{cases} 
\left( \frac{1-\lambda^2}{1+\lambda^2}, \frac{1-\lambda^2}{1+\lambda^2}x^2 \right), & (x, y) \neq (0, 0) \\
(0, 0), & (x, y) = (0, 0) 
\end{cases}$$

Then, for any $\lambda \neq 1$,

$$\lim_{x \to 0} f(x, \lambda x) \neq (0, 0)$$

So we do not have continuity at this point.

21. **Problem 2.1.21**

Define $f(x, y) := x^2 - y^2 + 2xy$. Then, $S = f^{-1}(-\infty, 0)$ is open since $f$ is continuous.
(i). Anticipating the result of (ii), note that $A \subset B(x, r) \implies \text{diam}(A) \leq 2r$, so that for all $x, y \in A$, $d(x, y) \leq 2r$, giving boundedness of $A$.

Conversely, if $d(x, y) \leq M$ for all $x, y \in M$, then given $\epsilon > 0$,

$$A \subset B(x, M + \epsilon)$$

(ii). This is tautological:

$$\text{diam}(A) = \sup_{x, y \in A} d(x, y) \leq \sup_{x, y \in B} d(x, y) = \text{diam}(B)$$

(iii). If $A = \{x\}$, then,

$$\text{diam}(A) = \sup_{x, y \in A} d(x, y) = d(x, x) = 0$$

Conversely, argue by contraposition. If $A \neq \{x\}$, then choose distinct $x, y \in A$. Then,

$$\text{diam}(A) \geq d(x, y) > 0$$

(iv). We have:

$$\text{diam}(A \cup B) - d(a, b) = \sup_{x, y \in A \cup B} d(x, y) - d(a, b)$$

$$= \sup_{x, y \in A \cup B} d(x, y) - d(x, a) + d(x, a) - d(a, b)$$

$$\leq \sup_{x, y \in A \cup B} d(a, y) + d(x, b)$$

$$\leq \sup_{y \in A} d(a, y) + \sup_{x \in B} d(x, b)$$

$$\leq \text{diam}(A) + \text{diam}(B)$$

(iv). If $A$ and $B$ are bounded, then the previous result shows that the union $A \cup B$ remains bounded. The general result follows by induction.
23. **Problem 2.1.23**

For all \( n \in \mathbb{N} \), there exists \( a_n \in A \) such that

\[ a - a_n < 1/n \]

Letting \( n \to \infty \), \( a_n \to a \), so \( a \in \overline{A} \).

24. **Problem 2.1.24**

By Urysohn’s Lemma, there exists a continuous separation of \( A \) and \( B \), denoted \( f : X \to [-1, 1] \). Then, consider for \( \epsilon > 0 \) sufficiently small:

\[ f^{-1}([-1, -1 + \epsilon)) := U \]
\[ f^{-1}((1 - \epsilon, 1]) := V \]

These sets are open by continuity of \( f \) and clearly \( A \subset U \), \( B \subset V \).

25. **Problem 2.2.1**

Note that \( d(x_n, x_m) \to 0 \) if and only if \( d_1(x_{n1}, x_{m1}) \to 0 \) and \( d_2(x_{n2}, x_{m2}) \to 0 \). Since each \( (X_i, d_i) \) is complete, we deduce that \( x_{n1} \to x_1 \in X_1 \) and \( x_{n2} \to x_2 \in X_2 \), so that \( (X_1 \times X_2, d_2) \) is complete.

26. **Problem 2.2.2**

\( F \) is complete if and only if it contains all of its limit points \( \iff \)

\( F \) is closed.

27. **Problem 2.2.3**

We have:
$||Tf - Tg||_\infty = \sup_{x \in I} \left| \int_0^x (x-t)(f(t) - g(t))dt \right|$

$\leq \sup_{x \in I} \left( \int_0^x |x-t|dt \right) ||f - g||_\infty$

$= \left( \int_0^1 (1-t)dt \right) ||f - g||_\infty$

$= \frac{1}{2} ||f - g||_\infty$

So that this is a contraction. Now, suppose that we have a fixed point of the above, so

$f(x) = x + \int_0^x (x-t)f(t)dt$

Taking the Laplace transform yields

$\tilde{f}(s) = \frac{1}{s^2} + \frac{\tilde{f}(s)}{s^2}$

$\Rightarrow \tilde{f}(s) = \frac{1}{s^2 - 1}$

$\Rightarrow f(x) = \sinh(x)$

And by Banach’s fixed point theorem, this must be unique.

**28. Problem 2.2.4**

We have:

$||Tf - Tg||_\infty = \sup_{x \in I} \left| \int_0^x f(t^2) - g(t^2)dt \right|$

$= \sup_{x \in I} \left| \frac{1}{2} \int_0^x \frac{f(u) - g(u)}{\sqrt{u}}du \right|$

$\leq \sup_{x \in I} \left( \frac{1}{2} \int_0^x \frac{du}{\sqrt{u}} \right) ||f - g||_\infty$

$= \sup_{x \in I} ||f - g||_\infty = k||f - g||_\infty$

Hence this is a contraction, so we have a unique fixed point. For every value in $k \in (0, 1)$, $T$ has a fixed point on the interval $[0, k]$. Hence we may take a sequence such that $Tf_n = f_n$ on $[0, 1 - 1/n]$. This is
bounded, and taking \( n \to \infty \), we may continuously extend this limit to the boundary.

29. Problem 2.2.5

(i). Set \( f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\} \) defined by \( f(x) = x/2 \). This is plainly a contraction, but there exists no fixed point.

(ii). Consider \(([1, \infty), d)\) with the induced metric space structure. This is trivially complete as a closed subspace of a complete space, and, we may set

\[
f(x) := x + 1/x
\]

So that

\[
|f(x) - f(y)| = \left|1 - \frac{1}{xy}\right| |x - y| < |x - y|
\]

However, \( f(x) = x \) implies that \( 1/x = 0 \), which is impossible, so we do not have a fixed point.

(iii). Define \( T f(x) := \int_a^x f(t) dt \) on the interval \([a, b]\), where \( b - a > 1 \). We then see

\[
\|T f - T g\|_\infty = \sup_{x \in [a, b]} \left| \int_a^x (f(t) - g(t)) dt \right| \\
\leq \sup_{x \in [a, b]} \int_a^x |f(t) - g(t)| dt \\
\leq (b - a) \|f - g\|_\infty
\]
And since $b - a > 1$, $T$ is not a contraction. However, we have that

$$T^n f = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt,$$

so

$$||T^n f - T^n g||_\infty = \sup_{x \in [a,b]} \left| \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} (f(t) - g(t)) dt \right|$$

$$\leq \sup_{x \in [a,b]} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} |f(t) - g(t)| dt$$

$$\leq \frac{(b-a)^n}{n!} ||f - g||_\infty$$

Hence, choosing $n$ large enough such that $\frac{(b-a)^n}{n!} < 1$, we see that $T^n$ becomes a contraction.

30. Problem 2.2.6

Suppose $x_n \to x \in X$ is $d_1$. Then $x_n \neq 0$ for any $n$, and $x \neq 0$. Also, $x_n$ is bounded below by some nonzero $x_*$, so that

$$\left| \frac{1}{x_n} - \frac{1}{x} \right| = \left| \frac{x - x_n}{x x_n} \right|$$

$$\leq \frac{|x - x_n|}{x_*^2} \to 0$$

As $n \to \infty$. Hence, $x_n \to x$ in $d_2$ as well. Similarly, if $x_n \to x$ in $d_2$,

$$|x - x_n| \leq \frac{|x - x_n|}{x x_n}$$

$$= d_2(x, x_n) \to 0$$

As $n \to \infty$. So $x_n \to x$ in $d_1$. Now, note that $(X, d_1)$ is not complete as the sequence $x_n = 1/n$ is Cauchy but not convergent in $X$. For $d_2$, the only problem point is 0. However, suppose $x_n \to 0$ and let $\epsilon > 0$. If $|x_n| < \epsilon$ for every $n > N$, then,

$$\left| \frac{1}{x_n} - \frac{1}{x_m} \right| > \frac{2}{\epsilon}$$

for all $n, m > N$. Hence, $x_n$ is not Cauchy.
31. Problem 2.2.7

We’ve already shown that this is indeed a metric from the previous section. It remains to show completeness. Suppose $x_n = (x_{nk})$ is Cauchy. Then, for every $k \in \mathbb{N}$,

$$\frac{|x_{nk} - x_{mk}|}{1 + |x_{nk} - x_{mk}|} \to 0$$

as $n, m \to \infty$. As $t \mapsto \frac{t}{t+1}$ is increasing, we deduce that $|x_{nk} - x_{mk}| \to 0$ in $\mathbb{R}$. As $\mathbb{R}$ is complete, we see that $x_{nk} \to x_k$ for each $k \in \mathbb{N}$.

Define $x := (x_k)$, the sequence consisting of the limits of each individual Cauchy sequence. By construction,

$$d(x, x_n) \to 0$$

as $n \to \infty$, so $x_n$ is convergent, yielding completeness.

32. Problem 2.2.8

(i). Note that

$$|d(x_n, y_n) - d(x_m, y_m)| = |d(x_n, y_n) - d(x_n, y_m) + d(x_n, y_m) - d(x_m, y_m)|$$

$$\leq |d(y_n, y_m) + d(x_n, x_m)|$$

$$\to 0$$

As $n, m \to \infty$, so that $d(x_n, y_n)$ is Cauchy.

(ii). Reflexive: $d(x_n, x_n) = 0$, so this is trivial.
Symmetric: $d(x_n, y_n) = d(y_n, x_n)$, so if either one tends to 0, so does the other.
Transitive: If $d(x_n, y_n), d(y_n, z_n) \to 0$, then

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \to 0$$

So this is an equivalence.
(iii). Similar to the first part,
\[ |d(x_n, y_n) - d(x'_n, y'_n)| = |d(x_n, y_n) - d(x_n, y'_n) + d(x_n, y'_n) - d(x'_n, y'_n)| \]
\[ \leq |d(y_n, y'_n) + d(x_n, x'_n)| \]
\[ \rightarrow 0 \]
As \( n \rightarrow \infty \).

(iv). Suppose \( \hat{d}([x_n], [y_n]) = 0 \). By definition,
\[ \lim_{n \rightarrow \infty} d(x_n, y_n) = 0 \quad \Rightarrow \quad x_n \sim y_n \]
\[ \Rightarrow \quad [x_n] = [y_n] \]
Proving nondegeneracy. Symmetry is trivial, and
\[ \hat{d}([x_n], [z_n]) = \lim_{n \rightarrow \infty} d(x_n, z_n) \]
\[ \leq \lim_{n \rightarrow \infty} d(x_n, y_n) + d(y_n, z_n) \]
\[ = \hat{d}([x_n], [y_n]) + \hat{d}([y_n], [z_n]) \]
So that \( \hat{d} \) is a metric.

(v). Define \( \varphi(x) := [(x_n)] \), where \( x_n \) is just the constant sequence \( x_n = x \) for every \( n \in \mathbb{N} \). Consider the association of \( X \) to \( \text{Im} \varphi \):
\[ \hat{d}(\varphi(x), \varphi(y)) = \lim_{n \rightarrow \infty} d([x], [y]) \]
\[ = d(x, y) \]
So that \( \varphi \) is an isometry.

(vi). Let \( [(x_n)] \in \hat{X} \). The sequence \( x_n \) is Cauchy, so for all \( \epsilon > 0 \), we can find \( m \in \mathbb{N} \) such that \( d(x_n, x_m) < \epsilon \). Set \( x := [x_m] \), the class of the constant sequence. By construction,
\[ \hat{d}([x_n], [y_n]) < \epsilon \]
So that \( X_0 \) is dense.
(vi). Let \([x]_n\) be Cauchy. For every \(n \in \mathbb{N}\), part (vi) shows that we can find \([y]_n \in X_0\) such that
\[
\hat{d}([x]_n, [y]_n) < 1/n
\]
Then for all \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that for all \(m, n > N\),
\[
\hat{d}([x]_m, [y]_n) \leq \hat{d}([x]_n, [y]_n) + \hat{d}([x]_n, [x]_m) < \frac{1}{n} + \epsilon
\]
Letting \(n \to \infty\), define \([y] := \lim_{n \to \infty} [y]_n\). The above shows
\[
\hat{d}([x]_m, [y]) < \epsilon
\]
And, as \(m \to \infty, \epsilon \to 0\), so that
\[
d([x]_m, [y]) \to 0 \text{ as } m \to \infty
\]
Implying \([x]_n \to [y] \in \hat{X}\), and \((\hat{X}, \hat{d})\) is complete.

33. Problem 2.3.1

Consider \(f_n = \frac{1}{x^n}, n \in \mathbb{N}\). Multiply this by the mollified characteristic function \(\eta_\epsilon * \chi_{[1/2, 1]}\), where \(\epsilon > 0\) is small enough such that
\[
\eta_\epsilon * \chi_{[1/2, 1]}|_{[0, 1/4]} \equiv 0
\]
Then, it is clear that \(\{f_n \cdot \eta_\epsilon * \chi_{[1/2, 1]}\}_{n \in \mathbb{N}}\) is a smooth unbounded family.

Now, let \(\epsilon > 0\). Setting \(\delta = \epsilon\), we see that for all \(\|f - g\|_\infty < \epsilon\),
\[
|Tf - Tg| \leq \left| \int_0^1 f(t) - g(t) dt \right| \\
\leq \|f - g\|_\infty < \epsilon
\]
So \(T\) is uniformly continuous.

\(^1\eta_\epsilon\) denotes the standard mollifier and \(\ast\) is the operation of convolution.
34. **Problem 2.3.2**

Argue by contraposition. Suppose there exists \( x \in X \) such that \( d(x, F_i) = 0 \) for every \( i \). Then, either \( x \in F_i \) or \( x \) is a limit point of the \( F_i \). As each \( F_i \) is closed, \( x \in \bigcap_{i \in I} F_i \neq \emptyset \), so we are done.

35. **Problem 2.3.3**

Suppose that

\[
\sup \{ d(x_0, y) \mid y \in B(x_0, r) \} \geq r
\]

Then for all \( \epsilon > 0 \), there exists some \( y \in B(x_0, r) \) such that \( r - \epsilon < d(x_0, y) \). But this implies

\[
B(x_0, r) \cap B(y, \epsilon) = \emptyset
\]

\[\implies d(x_0, y) > r\]

Which is a contradiction. Similarly, suppose

\[
\inf \{ d(x_0, y) \mid y \in X, d(x_0, y) > r \} \leq r
\]

Assume first this value is strictly less than \( r \). Then for \( \epsilon > 0 \) such that \( \inf \{ d(x_0, y) \mid y \in X, d(x_0, y) > r \} + \epsilon < r \), we can find \( y \in X \) with \( d(x_0, y) > r \) and

\[
d(x_0, y) < \inf \{ d(x_0, y) \mid y \in X, d(x_0, y) > r \} + \epsilon < r
\]

which is clearly impossible. Now assume \( \inf \{ d(x_0, y) \mid y \in X, d(x_0, y) > r \} = r \), so that given \( \epsilon < r \), we may find \( y \in X \) with \( d(x_0, y) < r + \epsilon \). Then,

\[
B(x_0, r) \cap B(y, \epsilon) \neq \emptyset
\]

\[\implies B(y, \epsilon) \subset B(x_0, r)\]

Which is again a contradiction. This immediately gives that our set must be countable, since there is a fixed minimum positive distance between every element.
36. Problem 2.3.4

Using compactness, choose convergent subsequences \( \{T^{n_k}(a)\}, \{T^{n_k}(b)\} \), \( k \in \mathbb{N} \). Given \( \epsilon > 0 \), we see that (assuming \( n_k > n_j \))
\[
\begin{align*}
\quad & d(T^{n_k}(a), T^{n_j}(a)) < \epsilon, \\
\quad & d(T^{n_k}(b), T^{n_j}(b)) < \epsilon \\
\implies & d(a, T^{n_k-n_j}(a)) < \epsilon, \\
\quad & d(b, T^{n_k-n_j}(b)) < \epsilon
\end{align*}
\]
So that if \( n_k - n_j = i \),
\[
\begin{align*}
\quad & d(T(a), T(b)) \leq d(T^i(a), T^i(b)) \\
\quad & \leq d(T^i(a), a) + d(a, b) + d(T^i(b), b) \\
\quad & < d(a, b) + 2\epsilon
\end{align*}
\]
As \( \epsilon > 0 \) is arbitrary, we see that \( d(T(a), T(b)) \leq d(a, b) \), implying \( d(T(a), T(b)) = d(a, b) \) so that \( T \) is an isometry. To show density of \( T(X) \), let \( \epsilon > 0 \). The above shows that \( d(a, T^k(a)) < \epsilon \) for some \( k \in \mathbb{N} \). Hence, \( T(X) \) is dense. By density, we may choose \( T x_n \to a \), where \( x_n \to c \). Then,
\[
\begin{align*}
\quad & d(a, Tc) \leq d(a, T x_n) + d(T x_n, T x) \\
\quad & = d(a, T x_n) + d(x_n, c) \\
\quad & \to 0
\end{align*}
\]
But \( d(a, Tc) \) is independent of \( n \), so we deduce that \( d(a, Tc) = 0 \), yielding \( a = Tc \). This completes the proof.

37. Problem 2.3.5

Suppose \( T x = x, T y = y \). Then
\[
d(x, y) = d(Tx, Ty) < d(x, y)
\]
Which is impossible, so the fixed point is unique. To show that such a fixed point exists in the first place, note that $f(x) := d(x, Tx)$ is continuous with respect to $x$. If $x \neq Tx$ for every $x$, then by compactness we are guaranteed that $f$ attains a minimum at some $x_0 \in X$, so that $f(x_0) \leq f(x)$ for every other $x \in X$. However,

$$d(Tx_0, T^2x_0) < d(x_0, Tx_0) \leq d(Tx_0, T^2x_0)$$

A contradiction. Hence there must exist some fixed point of $T$.

38. Problem 2.3.6

Without loss of generality we may suppose that $f_n$ is a decreasing sequence. Set $g_n := f_n - f$, so that $g_n$ is also decreasing and continuous. Given $\epsilon > 0$, $[0, \epsilon)$ is open in the subspace topology of $[0, \infty)$, so that

$$U_n := g_n^{-1}([0, \epsilon))$$

is open by continuity. By pointwise convergence, we deduce that $\bigcup_{n=1}^{\infty} U_n$ is an open cover of $X$, so that by compactness we may extract a finite subcover $\{U_1, \ldots, U_N\}$. As $g_n$ is decreasing, the $U_i$ are nested; in particular, we must have that $U_N = X$. However, this means that for all $n \geq N$,

$$d(g_n, 0) = d(f_n, f) < \epsilon$$

So that $f_n \to f$ uniformly.

39. Problem 2.3.7

Closure and boundedness are immediate. It remains to show equicontinuity. Let $f \in \mathcal{F}$. Then

$$|f(x) - f(y)| \leq \left( \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \right) \cdot |x - y|^{\alpha} \leq |x - y|^{\alpha}$$
So that given $\epsilon > 0$, we may take $\delta = \epsilon^{1/\alpha}$. Then for all $f \in \mathcal{F}$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \delta^\alpha = \epsilon$$

So this family is equicontinuous. By the Arzela-Ascoli theorem, we have compactness.

40. Problem 2.3.8

By the Arzela-Ascoli theorem, it suffices to show that $K$ is not equicontinuous. Consider the family $f_n(x) = x^n$. This family plainly consists of elements of $K$, but this family is not equicontinuous at $x = 1$. To see this, let $0 < \delta < 1/4$; employing Bernoulli’s inequality, choose $n > \frac{1}{2\delta}$:

$$|f_n(1 - \delta) - f(1)| = |(1 - \delta)^n - 1|$$

$$\geq |1 - n\delta - 1|$$

$$= |n\delta| > \frac{1}{2}$$

Hence we do not have equicontinuity, so $K$ cannot be compact.

41. Problem 2.3.9

Let $\epsilon > 0$. By equicontinuity, there exists $\delta$ such that

$$|f_n(x) - f_n(y)| < \epsilon/3 \text{ whenever } |x - y| < \delta$$

Letting $n \to \infty$ in the above,

$$|f(x) - f(y)| \leq \epsilon/3$$

By compactness, we may cover $X$ with finitely many $B(x_i, \delta)$, and for each $i$ we may find $N_i \in \mathbb{N}$ such that for all $n > N_i$,

$$|f_n(x_i) - f(x_i)| < \epsilon/3$$
Now choose $N := \max_i \{N_i\}$. For all $x \in X$, $x \in B(x_i, \delta)$ for some $i$.

Whenever $n > N$, we see that

$$ |f_n(x) - f(x)| \leq |f(x) - f(x_i)| + |f_n(x_i) - f_n(x)| + |f_n(x_i) - f(x_i)| $$

$$ < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon $$

So that $f_n \to f$ uniformly.

42. **Problem 2.3.10**

This is clearly bounded. It remains to show equicontinuity. Note that

$$ |f'(t)| = \left| \frac{\cos(\sqrt{t + 4\pi^2 n^2})}{2\sqrt{t + 4\pi^2 n^2}} \right| $$

$$ \leq \frac{1}{4n\pi} $$

$$ \leq \frac{1}{4\pi}, \text{ for } n \geq 1 $$

Then, using the mean value theorem we clearly have

$$ |f_n(x) - f_n(y)| \leq \frac{1}{4\pi} |x - y| $$

So this is an equicontinuous family. Similarly, the above immediately gives that

$$ |f_n(t)| \leq \frac{|t|}{4\pi n} $$

So, holding $t$ fixed, $f_n(t) \to 0$ as $n \to \infty$ implying that $f_n \to 0$ pointwise.

It remains to show that compactness fails. Choose any subsequence $f_{n_k}$, and consider

$$ f_{n_k}((2n_k + 1/4)\pi^2) = \sin \left( \sqrt{2n_k \pi^2 + \pi^2/4 + 4n_k^2 \pi^2} \right) $$

$$ = \sin(\pi \sqrt{4n_k^2 + 2n_k + 1/4}) $$

$$ = \sin \left( \pi(2n_k + 1/2) \right) $$

$$ = 1, \text{ for all } k $$
Hence, no subsequence converges to zero, so this set is certainly not compact.

43. **Problem 2.3.11**

For all $f \in \mathcal{K}$, let $\epsilon > 0$. For $\delta = \epsilon / M,$

$$|f(x) - f(y)| = \frac{|f(x) - f(y)|}{|x - y|} |x - y|$$

$$\leq M|x - y|$$

$$< \epsilon$$

So this is an equicontinuous family.

44. **Problem 2.3.12**

If $f$ is lower semicontinuous, then for all $\epsilon > 0$ there exists $\delta$ such that

$$f(x) - \epsilon < f(x_n) \text{ when } d(x, x_n) < \delta$$

Taking the infimum over all $n$ in the above, this implies

$$f(x) - \epsilon \leq \inf_{m> n} f(x_n)$$

$$\implies f(x) - \epsilon \leq \lim_{n \to \infty} \inf f(x_n)$$

Letting $\epsilon \to 0$, we see that $f(x) \leq \lim \inf_{n \to \infty} f(x_n)$. For the converse, we argue by contraposition. Then, for all $\delta > 0$ there exists some $\epsilon > 0$ and $x_n$ such that,

$$f(x) - \epsilon \geq f(x_n) \text{ and } d(x, x_n) < 1/n$$

Then, choose such an $x_n$ for each $n$. Obviously $x_n \to x$, but

$$f(x) - \epsilon \geq \lim_{n \to \infty} \inf f(x_n)$$

$$\implies f(x) > \lim_{n \to \infty} \inf f(x_n)$$

Whence the result.
Problem 2.3.13

(i). Fix $n \in \mathbb{N}$ and let $\epsilon > 0$. We can find $\delta$ such that $f(x) > f(y) - \epsilon$ whenever $d(x, y) < \delta$. Hence,

$$g_n(x) = \inf_{y \in K} \left( f(y) + nd(x, y) \right)$$

$$< \inf_{y \in K} \left( f(x) + \epsilon + nd(x, y) \right)$$

$$= f(x) + \epsilon + \inf_{y \in K} nd(x, y)$$

$$= f(x) + \epsilon$$

Letting $\epsilon \to 0$, we find that $g_n(x) \leq f(x)$ for every $n$. We also have:

$$g_{n+1}(x) = \inf_{y \in K} f(y) + (n + 1)d(x, y)$$

$$\geq \inf_{y \in K} f(y) + nd(x, y) + \inf_{y \in K} d(x, y)$$

$$= \inf_{y \in K} f(y) + nd(x, y)$$

$$= g_n(x)$$

Hence the sequence $g_n$ is increasing an bounded above by $f$, so we deduce that is converges pointwise to $f$.

(ii). This follows from Theorem 2.2.25 in the book, since the set of discontinuities is of first category. By Theorem 2.2.27, every complete metric space is a Baire space, so that upon taking the complement the set of continuous points must be dense.
We have:

\[ v(t) \leq c + \int_a^t v(s)u(s)ds \]

\[ \Rightarrow \frac{v(t)}{c + \int_a^t v(s)u(s)ds} \leq 1 \]

\[ \Rightarrow \frac{u(t)v(t)}{c + \int_a^t v(s)u(s)ds} \leq u(t) \]

\[ \Rightarrow \frac{d}{dt} \log \left( c + \int_a^t v(s)u(s)ds \right) \leq u(t) \]

\[ \Rightarrow \log \left( c + \int_a^t v(s)u(s)ds \right) \leq K + \int_a^t u(s)ds \]

\[ \Rightarrow c + \int_a^t v(s)u(s)ds \leq C \exp \left( \int_a^t u(s)ds \right) \]

Now, setting \( t = a \) in the above, we find that \( c = C \). Also, as we are given \( v(t) \leq c + \int_a^t v(s)u(s)ds \), we deduce from the last line of the above

\[ v(t) \leq c \exp \left( \int_a^t u(s)ds \right) \]

Now if \( c = 0 \) in the above, we get that \( v(t) \leq 0 \), and since \( v \) is assumed nonnegative, \( v \equiv 0 \).

Assume now that \( u_1 \) and \( u_2 \) satisfy the initial value problem

\[ u_i(t) = u_i(a) + \int_a^t f(s, u_i(s))ds, \ i = 1, 2 \]

Then we consider \( |u_1 - u_2| \). This is clearly nonnegative, and

\[ |u_1 - u_2| \leq \int_a^t |f(s, u_1(s))ds - f(s, u_2(s))|ds \]

\[ \leq \int_a^t K|u_1 - u_2|ds \]

where we’ve used that \( f \) is Lipschitz. Applying Grönwall’s to the above with \( v(t) := |u_1 - u_2| \) and \( u \equiv K \), we see that \( c = 0 \) so that

\[ |u_1 - u_2| = 0 \Rightarrow u_1 = u_2 \]
yielding uniqueness.

47. Problem 2.3.15

We proceed by contradiction. If the result is false, then we may find $A_n$ such that $\text{diam}(A_n) < 1/n$ and $A_n$ is not contained in any element of our open cover for each $n$. Choose $x_n \in A_n$, and using compactness we have a convergent subsequence $x_{n_k} \to x$. We have that $x \in U_i$ for some $i$, and by openness, we can find $\epsilon > 0$ such that $B(x, \epsilon) \subset U_i$.

Choosing $n > 2/\epsilon$, we can find $x_n \in B(x, \epsilon/2)$. Then for all $a \in A_n$,
\[
\begin{align*}
d(x, a) & \leq d(a, x_n) + d(x, x_n) \\
& \leq \text{diam}(A_n) + d(x, x_n) \\
& < \epsilon/2 + \epsilon/2 = \epsilon
\end{align*}
\]
So that $A_n \subset B(x, \epsilon) \subset U_i$, contrary to the choice of the $A_n$.

48. Problem 2.3.16

We first show the desired equality. Without loss of generality, assume $\delta(A, B) = \sup_{a \in A} d(a, B)$. Then for every $r > \delta(A, B)$, by definition we have that
\[
A \subset V_r(B)
\]
Taking the infimum over all $r$, we see that
\[
\delta(A, B) \leq \inf_{r > 0} \{A \subset V_r(B), \ B \subset V_r(A)\}
\]
Now if $r = \delta(A, B)$, we have that
\[
A \subset V_r(B), \ B \subset V_r(A)
\]
Hence by definition of infimum, we must have that
\[
\inf_{r > 0} \{A \subset V_r(B), \ B \subset V_r(A)\} \leq \delta(A, B)
\]
So indeed we have equality. Now we proceed to show that $\delta$ is a metric.

If $\delta(A, B) = 0$, then $A \subset B$ and $B \subset A$, so $A = B$. Symmetry is a tautology. Let $C \in \mathcal{K}$, and $c \in C$. To show the triangle inequality, suppose again without loss of generality that $\delta(A, B) = \sup_{a \in A} d(a, B)$.

Then, let $c_n$ be such that $d(a, c_n) \to d(a, C)$. 

$$\delta(A, B) = \sup_{a \in A} d(a, B)$$

$$= \sup_{a \in A} \inf_{b \in B} d(a, b)$$

$$\leq \sup_{a \in A} \inf_{b \in B} d(a, c_n) + d(b, c_n)$$

$$\leq \sup_{a \in A} d(a, C) + \inf_{b \in B} d(b, c_n)$$

$$= \delta(A, C) + d(c_n, B)$$

$$\leq \delta(A, C) + \delta(B, C)$$

So we have the triangle inequality, and hence $\delta$ is a metric. Let us show now that this space is complete. Let $A_n$ be a Cauchy sequence with respect to our Hausdorff metric. Without loss of generality (by passing to a subsequence if need be), we may assume that $\delta(A_n, A_{n+1}) < 2^{-n}$. We may select a sequence $(x_n)_{n \geq N}$ for every $N \in \mathbb{N}$ such that $x_n \in A_n$ and $d(x_n, x_{n+1}) < 2^{-n}$. Obviously this sequence is Cauchy by construction, and by completeness of our underlying space, converges to some $x \in X$. Also,

$$d(x_n, x_m) \leq \sum_{k=n}^{m} d(x_k, x_{k+1})$$

$$< \frac{1}{2^n} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^m} \right)$$

So that, letting $m \to \infty$ in the above, $d(x, x_n) < \frac{1}{2^{n-1}}$. We may now define the limit of our sequence $A_n$ to be the set $A$ consisting of all limits $x$ of sequences $x_n$ such that $x_n \in A_n$ for every $n \in \mathbb{N}$. $A$ is nonempty by the previous paragraph, and we may further arrange by
passing to a subsequence that $d(x_n, x_{n+1}) < 2^{-n}$. Given any $n \in \mathbb{N}$, there exists $x_n \in A_n$ such that $d(x, x_n) < \frac{1}{2^n-1}$.

Now let $\epsilon > 0$. Find $N$ large enough that $\epsilon > 2^{-N}$. For every $x \in A$, we may find $x_n$ such that $d(x, x_n) < 2^{-n+1}$. Now choose $n \geq N + 1$. For every $x \in A$, $d(x_n, x) < 2^{-n+1} < \epsilon$, so that $\delta(A, A_n) < \epsilon$ and $A \subseteq A_n$. Then $A$ is certainly closed and bounded, and as a closed subset of a compact set in a Hausdorff space, $A$ is compact. Hence we conclude that $K$ is compact under the Hausdorff metric.

Now, we proceed to prove the desired equality. Proceed by induction on $n$, with the base case being for $n = 2$. Then,

$$\delta(A \cup A', B \cup B') = \sup_{a \in A \cup A'} d(a, B \cup B')$$

$$= \sup_{a \in A \cup A'} \inf_{b \in B \cup B'} d(a, b)$$

Then the supremum is achieved by restricting to either $A$ or $A'$, and similarly for the infimum, $B$ or $B'$. Hence we may take the maximum over these; that is

$$\delta(A \cup A', B \cup B') = \max\{\delta(A, B), \delta(A, B'), \delta(A', B), \delta(A', B')\}$$

Proving the base case. Now assume the inductive hypothesis for all integers less than $n$, and take $A^{n-1} = \bigcup_{i=1}^{n-1} A_i$, $B^{n-1} = \bigcup_{i=1}^{n-1} B_i$, so

$$\delta\left(\bigcup_{i=1}^{n} A_i, \bigcup_{i=1}^{n} B_i\right) = \delta(A^{n-1} \cup A_n, B^{n-1} \cup B_n)$$

$$= \max\{\delta(A^{n-1}, B_n), \delta(A_n, B^{n-1}), \delta(A^{n-1}, B^{n-1}), \delta(A_n, B_n)\}$$

$$= \max\{\max_{i=1, \ldots, n-1}\{\delta(A_i, A_n)\}, \max_{i=1, \ldots, n-1}\{\delta(A_n, B_i)\},$$

$$\max_{i=1, \ldots, n-1}\{\delta(A_i, B_i)\}, \delta(A_n, B_n)\}$$

$$= \max_{i=1, \ldots, n}\{\delta(A_i, B_i)\}$$
Whence the result. Now suppose that $F$ is a contraction. We see:

$$
\delta(F(A), F(B)) = \sup_{a \in A} \inf_{b \in B} d(F(a), F(b)) \\
= \sup_{a \in A} \inf_{b \in B} rd(a, b) \\
= r \delta(A, B)
$$

So this is also a contraction. Now define $F$ as in the book. By the above two results,

$$
\delta(F(A), F(B)) = \max_i \{ \delta(F_i(A), F_i(B)) \} \\
\leq \max_i \{ r_i \} \max_i \{ \delta(A, B) \} \\
= \max_i \{ r_i \} \delta(A, B)
$$

Since each $r_i < 1$, $\max_i \{ r_i \} < 1$, so this is also a contraction. By Banach’s fixed point theorem, there exists a unique compact set $K \in \mathcal{K}$ such that $F(K) = K$. That is,

$$
\bigcup_{i=1}^{n} F_i(K) = K
$$

Net, let $X = [0, 1]$, $F_1(x) = x/3$ and $F_2(x) = (2 + x)/3$. These are both clearly contractions, and by the proof of the Banach fixed point theorem, the sequence $F^n([0, 1])$ converges to the desired fixed point, which in this case yields the Cantor set.

49. Problem 2.4.1

Assume $B$ is not connected. Then there exists open $U, V$ such that $U \cap B \neq \emptyset$, $V \cap B \neq \emptyset$, $U \cap V \cap B = \emptyset$, and $B \subset U \cup V$. Since $A \subset B$, we also see that $A \cap U \cap V = \emptyset$. It remains to show that $A \cap U$ and $A \cap V$ are nonempty. Suppose then that $A \subset U^c$. Since $U^c$ is closed, we deduce that $\overline{A} \subset U^c$, and as $B \subset \overline{A}$, $B \subset U^c$ as well. This is a clear
contradiction, as $U \cap B \neq \emptyset$. Hence, $A \cap U \neq \emptyset$, and by symmetry we also deduce that $A \cap V \neq \emptyset$.

However, this gives that $U$ and $V$ disconnect $A$ as well. Taking contrapositives, we obtain the result.

50. Problem 2.4.2

$\mathbb{R}^2$ is connected, so suppose that $U$ is clopen. Then $U \cup U^c$ constitutes a disconnect of $\mathbb{R}^2$, so we deduce that $U = \mathbb{R}^2$ or $U = \emptyset$.

51. Problem 2.4.3

$E$ is path connected since in the union of the two sets is such that every point can be path connected to the point $(1, 0)$, hence to every other point of $E$. This also gives that the connected components consist of $E$ itself.

For $F$, the set of singletons $\{(1+1/n, 0)\}$ is disconnected as it inherits the discrete topology. Hence each of the singletons is a connected component, and the other component consists only of the other open ball in the union.

52. Problem 2.4.4

Associate $e_{ij}$, the matrix with 1 in the $(i, j)$ spot and 0 elsewhere the basis vector $e_{i+n(j-1)}$ in $\mathbb{R}^{n^2}$. Extend this by linearity and this is the desired identification. We may endow $\mathbb{R}^{n^2}$ with the Euclidean metric. Consider now the determinant function $\det : GL(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$. Since the image is disconnected, it remains to show that the determinant function is a continuous function under the Euclidean metric in order to prove the general linear group is also not connected.
Let \( M = (a_{ij})_{n \times n} \) and \( M' = (a'_{ij})_{n \times n} \). Recall

\[
\det M = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}, \det M' = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a'_{i,\sigma_i}
\]

where \( S_n \) is the set of all permutations of \( \{1, 2, \cdots, n\} \) and \( \sigma_i = \sigma(i) \).

Define \( m = \max_{1 \leq i, j \leq n} \{|a_{ij}|, |a'_{ij}|\} \). Then

\[
| \det M - \det M' | = \left| \sum_{\sigma \in S_n} \text{sgn}(\sigma) \left( \prod_{i=1}^{n} a_{i,\sigma_i} - \prod_{i=1}^{n} a'_{i,\sigma_i} \right) \right|
\]

\[
\leq \sum_{\sigma \in S_n} \left| \prod_{i=1}^{n} a_{i,\sigma_i} - \prod_{i=1}^{n} a'_{i,\sigma_i} \right|
\]

\[
= \sum_{\sigma \in S_n} \left| a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n} - a'_{1,\sigma_1} a'_{2,\sigma_2} \cdots a'_{n,\sigma_n} \right|
\]

\[
= \sum_{\sigma \in S_n} \left| (a_{1,\sigma_1} - a'_{1,\sigma_1}) a_{2,\sigma_2} a_{3,\sigma_3} \cdots a_{n,\sigma_n} + a'_{1,\sigma_1} (a_{2,\sigma_2} - a'_{2,\sigma_2}) a_{3,\sigma_3} \cdots a_{n,\sigma_n} + \cdots + a'_{1,\sigma_1} a'_{2,\sigma_2} a'_{3,\sigma_3} \cdots a'_{n-1,\sigma_{n-1}} (a_{n,\sigma_n} - a'_{n,\sigma_n}) \right|
\]

\[
\leq \sum_{\sigma \in S_n} \sum_{i=1}^{n} m^{n-1} |a_{i,\sigma_i} - a'_{i,\sigma_i}|
\]

Then, given any \( \varepsilon > 0 \), set \( \delta = \frac{\varepsilon}{2nn!m^{n-1}} \). Then, whenever

\[
\| M - M' \| < \delta.
\]

We have that \( |a_{i,j} - a'_{i,j}| < \delta \) for all \( 1 \leq i, j \leq n \) and hence

\[
| \det M - \det M' | \leq \sum_{\sigma \in S_n} \sum_{i=1}^{n} m^{n-1} \delta = nn!m^{n-1}\delta = \frac{\varepsilon}{2} < \varepsilon.
\]

Hence we see that \( \det \) is continuous, implying that \( \text{GL}(n, \mathbb{R}) \) is also disconnected.
53. Problem 2.4.5

Let \( c \in A \cap B \). Given any \( a \in A, b \in B \), there exists a path \( f(t) \) from \( a \) to \( c \) and another path \( g(t) \) from \( c \) to \( b \). Concatenating, we get a path from \( a \) to \( b \). Since \( a \) and \( b \) were arbitrary, we conclude that the union remains path connected.

54. Problem 2.4.6

Given \( f(a), f(b) \in f(E) \), we may find a path in \( g(t) \) in \( E \) between \( a \) and \( b \) since \( E \) is path connected. Now compose with \( f \), and we see that \( f \circ g \) is continuous as the composition of continuous functions, and becomes a path from \( f(a) \) to \( f(b) \).

55. Problem 2.4.7

The Cantor set is a closed subset of a compact space (namely, the closed unit interval), and since \([0,1]\) is Hausdorff, we deduce that the Cantor set is also compact\(^2\). By construction of the Cantor set \( K \), \([0,1]\setminus K \) is the union of the open middle thirds intervals removed in each iteration. This is clearly a countable union.

Note that if \( \mu \) denotes Lebesgue measure, \( \mu(C^n) = (2/3)^n \) where \( C^n \) denotes the \( n \)th iterate of the sequence tending to the Cantor set. \( C^n \) is a decreasing nested sequence, and as a subspace of a finite measure space, we see

\[
\mu(K) = \mu(\lim_{n \to \infty} C^n) = \lim_{n \to \infty} \mu(C^n) = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0
\]

\(^2\)Alternatively, use the result of Problem 2.3.16 to note that this was already proved.
So the Cantor set has measure 0, and since \( \mathcal{K} \cup \mathcal{K}^c = [0,1] \) and 
\( \mu([0,1]) = 1 \), we deduce that \( \mu(K^c) = 1 \). Finally, we also see that 
\( K \) is trivially disconnected, since if not, there would exist \( \epsilon > 0 \) such 
that \( K \) contains some interval of length at least \( \epsilon \). But \( \mu(K) = 0 \), so 
this is clearly impossible. Hence, \( K \) is completely disconnected, implying the 
connected component of each \( x \in K \) is just the singleton set \{\( x \}\).

56. Problem 2.4.8

Endow the set \{0, 2\} with the discrete topology and let \{0, 2\}^N be 
given the product topology. Then, noting that the Cantor set \( K \) inherits 
the discrete topology from \([0,1]\), it is obvious that \( K \) is homeomor-
phic to \{0, 2\}^N by the map sending any \( c \in K \) to the ternary decimal 
expansion as a sequence in \{0, 2\}^N (call this map \( h : K \to \{0, 2\}^N \)). 
We will decompose the given map in the book as a composition of con-
tinuous functions to deduce that we have continuity. Firstly, note that 
surjectivity is obvious. Given \((x,y) \in [0,1] \times [0,1] \), put \( x \) and \( y \) into 
their binary expansions. Then, if \( x = 0.a_1a_2\ldots \) and \( y = 0.b_1b_2\ldots \), the 
preimage in the Cantor set is the number whose ternary expansion is 
0.(2b_1)(2a_1)(2b_2)(2a_2)\ldots.

We now proceed to show continuity. Consider first the map \( g : \{0, 2\}^N \to [0,1] \) defined by 
\[
g((a_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} a_n \frac{1}{2^n}\]
Give \([0,1]\) the induced metric topology from \( \mathbb{R} \) and let \( \epsilon > 0 \). There 
exists \( N \in \mathbb{N} \) such that 
\[
\sum_{n \geq N+1} \frac{1}{2^n} < \epsilon
\]
Let \( c \in [0,1] \) and write
\[
c = \sum_{n=1}^{\infty} \frac{c_n}{2^n}
\]
in its binary expansion. If \( p_i : \{0,2\}^N \to [0,1] \) such that \( p_i((a_n)_{n \in \mathbb{N}}) = a_i/2 \), consider the set
\[
U := p_1^{-1}(c_1) \cap \cdots \cap p_N^{-1}(c_N)
\]
As the intersection of finitely many open sets\(^3\), this set remains open. We also see that given any \((a_n) \in U\), by construction,
\[
|g(a_n) - c| = \sum_{n=1}^{\infty} \frac{|a_n - c_n|}{2 \cdot 2^n} = \sum_{n \geq N+1} \frac{|a_n - c_n|}{2 \cdot 2^n} \leq \sum_{n \geq N+1} \frac{1}{2^n} < \epsilon
\]
Hence, \( g(U) \subset B_\epsilon(c) \), proving continuity of \( g \). We may now consider a continuous bijection \( \ell : \{0,2\}^N \to \left(\{0,2\}^N\right) \times \left(\{0,2\}^N\right) \) by mapping \((c_n) \mapsto ((c_{2n+1}), (c_{2n}))\). Continuity is trivial as both sets are given the product topology.

Finally, we prove continuity of \( f \). We may factor \( f \) in the following fashion:
\[
K \xrightarrow{h} \{0,2\}^N \xrightarrow{\ell} \left(\{0,2\}^N\right) \times \left(\{0,2\}^N\right) \xrightarrow{g \times g} [0,1] \times [0,1]
\]
Since each of the above maps has already been proved continuous, we deduce that the composition is continuous. But the composition is precisely \( f \), so the proof is complete.

\(^3\)Note that these sets are open as each preimage is of the form of a singleton set crossed with infinitely many other \( \{0,2\} \), which by definition is open in the product topology.
57. **Problem 2.5.1**

Define

\[ H(x, t) := \frac{(1-t)f(x) + tx}{||(1-t)f(x) + tx||} \]

Since \( f \) is continuous and \( f(x) \neq -x \) for any \( x \), the above is well defined and also continuous as the quotient of continuous functions. For all \( t \in [0, 1] \), by construction \( ||H(x, t)|| = 1 \), so this is a path in \( S^n \); clearly \( H(x, 0) = f(x) \) and \( H(x, 1) = x \). Hence, we have a homotopy from \( f \) to the identity.

58. **Problem 2.5.2**

Suppose first that \( \mu \sim \nu \). Then,

\[ \mu \ast \hat{\nu} \sim \nu \ast \hat{\nu} \]

\[ \sim e_x \]

So that by transitivity, we deduce that \( \mu \ast \hat{\nu} \sim e_x \). Conversely,

\[ \mu \sim \mu \ast e_y \]

\[ \sim \mu \ast (\hat{\nu} \ast \nu) \]

\[ \sim (\mu \ast \hat{\nu}) \ast \nu \]

\[ \sim e_x \ast \nu \]

\[ \sim \nu \]

And again by transitivity, we deduce that \( \mu \sim \nu \), whence the result.

59. **Problem 2.5.3**

By definition, it is clear that for any nonconstant closed loop,

\[ (f \ast f) \ast f \neq f \ast (f \ast f) \]

so choosing any nonconstant closed loop will satisfy the first condition. For the case of equality, we then deduce that our map must be constant.
Hence, setting \( g \equiv (1,0) \in \mathbb{R}^2 \),

\[
(g \ast g) \ast g = g \ast (g \ast g)
\]

60. Problem 2.5.4

When \( a = 1 \) or \( 0 \), \( g \equiv f \), so this case is done. Suppose now that \( 0 < a < 1 \). We only need show that \( H \) established a free homotopy between \( f \) and \( g \), since in such a case it is obvious that \( f^* = f^* \).

We first note that \( H(s,0) = f(s) \) and \( H(s,1) = g(s) \). It remains to show continuity; whenever \( s \neq 1 - ta \), \( H \) is obviously continuous, so it suffices only to show that \( H \) is continuous at \( s = 1 - ta \). We see

\[
\lim_{s \to (1-ta)^-} H(s,t) = \lim_{s \to (1-ta)^-} f(s + ta) = f(1)
\]

\[
\lim_{s \to (1-ta)^+} H(s,t) = \lim_{s \to (1-ta)^+} f(ta + s - 1) = f(0)
\]

Since \( f \) is given as a closed path, \( f(0) = f(1) \), so that our left and right limits agree, yielding continuity. This completes the problem.

61. Problem 2.5.5

We define a standard straight line homotopy:

\[
H(x,t) := (1 - t)f_0(x) + tf_1(x)
\]

By assumption, this is well defined for every \( t \) and \( x \) and clearly continuous, and we are done.

62. Problem 2.5.6

Reflexivity is clear since we may take \( f \equiv g \equiv id_X \). Symmetry is tautological. Now, let \( X \simeq Y \) and \( Y \simeq Z \). We have maps

\[
f : X \to Y, \quad h : Y \to Z
\]

\[
g : Y \to X, \quad i : Z \to Y
\]
Then, 
\[(g \circ i) \circ (h \circ f) = g \circ (i \circ h) \circ f\]
\[\simeq g \circ \text{id}_Y \circ f\]
\[= g \circ f \simeq \text{id}_X\]
And
\[(h \circ f) \circ (g \circ i) = h \circ (f \circ g) \circ i\]
\[\simeq h \circ \text{id}_Y \circ i\]
\[= h \circ i \simeq \text{id}_Z\]
So that \(X \simeq Z\), proving transitivity. Therefore \(\simeq\) is an equivalence relation.

63. **Problem 2.5.7**

(i). Note that \(X\) and \(Y\) are not homeomorphic since we may remove the point \((2, 0)\) from \(Y\) and it remains connected, which is a property that \(X\) does not have. Now consider the inclusion \(i : X \hookrightarrow Y\) and the projection \(r : Y \to X\) defined by \(r(Y \setminus X) = (1, 0)\) and \(r|_X \equiv \text{id}_X\).

We see that \(r \circ i = \text{id}_X\) by construction, and, taking the straight line homotopy
\[(1 - t)i \circ r + t \cdot \text{id}_Y\]
we find that \(i \circ r \simeq \text{id}_Y\), as desired.

(ii). These two are obviously not homeomorphic as you can remove any point from \(\mathbb{R}^2 \setminus \{0\}\) and it remains path connected, which is not possible for \(S^1\). We again take the inclusion \(i : S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}\). Then, define \(r : \mathbb{R}^2 \setminus \{0\} \to S^1\) by
\[r(y) := \frac{y}{||y||}\]
This is well defined and continuous, and we also immediately have that $r \circ i = \text{id}_{S^1}$. For $i \circ r$, again we may merely take the straight line homotopy to the identity:

$$(1 - t)i \circ r + t \cdot \text{id}_{\mathbb{R}^2 \setminus \{0\}}$$

64. Problem 2.5.8

Suppose first that $X$ is contractible. Let $X \to \{\text{pt}\}$, $\{\text{pt}\} \to X$. Then the composition $X \to \{\text{pt}\} \to X$ is homotopic to the identity by the given (as this is merely the constant map $x \mapsto \text{pt}$), and, the composition $\{\text{pt}\} \to X \to \{\text{pt}\}$ is already the identity map. Hence, by definition, $X \simeq \{\text{pt}\}$.

Conversely, suppose $X \simeq \{\text{pt}\}$. Then the map $X \to \{\text{pt}\} \to X$ is homotopic to the identity. But as already noted, this map is precisely the constant map, so that $\text{id}_X \simeq \text{pt}$, and $X$ is contractible.

From this we may deduce that every convex set is contractible. Let $x \in X$ and denote by $x$ the constant map sending any other $y$ to $x$. We may take a straightline homotopy, which is well defined by definition of convexity,

$$H(x, t) := (1 - t)x + t \cdot \text{id}_X$$

Which shows that the identity is homotopic to the constant map, so by definition $X$ is contractible, and we are done.