NOTES IN COMMUTATIVE ALGEBRA: PART 1

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1. Results/Definitions of Ring Theory

It is in this section that a collection of standard results and definitions in commutative ring theory will be presented. For the rest of this paper, any ring R will be assumed commutative with identity. We shall also use "$=$" and "$\cong$" (isomorphism) interchangeably, where the context should make the meaning clear.

1.1. **The Basics.**

Definition 1.1. A maximal ideal is any proper ideal that is not contained in any strictly larger proper ideal. The set of maximal ideals of a ring R is denoted m-Spec(R).

Definition 1.2. A prime ideal $\mathfrak{p}$ is such that for any a, $b \in R$, $ab \in \mathfrak{p}$ implies that a or $b \in \mathfrak{p}$. The set of prime ideals of R is denoted Spec(R).

Definition 1.3. The radical of an ideal I, denoted $\sqrt{I}$, is the set of $a \in R$ such that $a^n \in I$ for some positive integer n.

Definition 1.4. A *primary* ideal $\mathfrak{p}$ is an ideal such that if $ab \in \mathfrak{p}$ and $a \notin \mathfrak{p}$, then $b^n \in \mathfrak{p}$ for some positive integer n.

In particular, any maximal ideal is prime, and the radical of a primary ideal is prime.

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Definition 1.5. The notation $(R, \mathfrak{m}, k)$ shall denote the local ring R which has unique maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$.

Example 1.6. Consider the set of smooth functions on a manifold M. Define an equivalence relation by considering f and g related at a point $p \in M$ if there exists a neighborhood U of p such that $f|_U = g|_U$. Then, let $[f]$ denote the class of f, which is referred to as its *germ*. The set of germs at p is a commutative local ring, with $[f] + [g] := [f + g]$, and $[f][g] := [fg]$. The maximal ideal is precisely the set of functions such that $f(p) = 0$.

Example 1.7. Given a commutative C^*-algebra A, set $X = \text{m-Spec}(A)$. Then $A/J = \mathbb{C}$ for all $J \in X$ by the Gelfand-Mazur theorem. Hence there exists a naturally defined homomorphism

$$\pi_J : A \to \mathbb{C}$$

Now, to each $a \in A$ associate a function $\widehat{a}$ on X defined by

$$\widehat{a}(J) = \pi_J(A)$$

Then $\widehat{a}$ is called the *Gelfand Transform* of $a \in A$. It is due to a result of Gelfand-Naimark that if A is unital, the Gelfand transform is an isomorphism of A onto the space of continuous functions on X.

Definition 1.8. The Jacobson radical $J(R)$ is the intersection of all maximal ideals of the ring R.

The Jacobson radical has a nice characterization:

Proposition 1.9. *If $a \in J(R)$, then $1+a$ is a unit. Moreover, $J(R) = \{x \in R \mid 1 + Rx \subset R^\times\}$.*

Proof. Suppose first that $a \in J(R)$. Then, $1+ax \notin \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$, since else $1 = m-ax \in \mathfrak{m}$, a contradiction. Hence, $(ax+b) = R$ so that $r(1+ax) = 1$ for some $r \in R$.

Conversely, argue by contraposition. If $a \notin J(R)$, then we can find a maximal ideal $\mathfrak{m}$ such that $a \notin \mathfrak{m}$ so that $(a)+\mathfrak{m} = R$. Thus there exists $r \in R$ and $m \in \mathfrak{m}$ such that $ra+m = 1$. But then $m = 1-ra \in 1+Ra$, and m is not a unit, so we are done. □

Example 1.10. For any local ring $(R, \mathfrak{m}, k)$, $J(R) = \mathfrak{m}$, and the set of units $R^\times$ is merely $R \backslash \mathfrak{m}$. Indeed, R is local if and only if $1+\mathfrak{m}$ consists entirely of units.

Definition 1.11. An R-module M will be called finitely generated if there is a finite set $\{x_i\}$ such that given $x \in M$ there exists $r_i \in R$ for which $x = r_1 x_1 + \cdots + r_n x_n$. The category of all finitely generated R-modules will be denoted by $\mod R$.

Example 1.12. In the above, if R is a field, then we merely have a vector space of dimension $n < \infty$.

The following is used to prove a fundamental result in commutative algebra known as Nakayama's Lemma.

Theorem 1.13. *Let M be a finitely generated R-module with $\mathfrak{a}$ an ideal of R. If $\phi : M \to M$ is an R-module homomorphism such that $\phi(M) \subset \mathfrak{a}M$ then there exists a monic polynomial $p(x) \in \mathfrak{a}[x]$ such that $p(\phi) = 0$. More precisely, there exist $a_i \in \mathfrak{a}$ such that*

$$\phi^n + a_1\phi^{n-1} + \cdots + a_{n-1}\phi + a_n = 0 \tag{1.1}$$

Proof. Choose a generating set $\{x_1, \ \ldots, \ x_n\}$. Then, for each i, we have that $\phi(x_i) = \sum_j a_{ij}x_j$ where each $a_{ij} \in \mathfrak{a}$.

Subtracting, the above can be stated concisely as $\sum_j(\delta_{ij}\phi - a_{ij})x_j = 0$ (δ_{ij} denotes Kronecker delta). Then, if A is defined to be the matrix with entries $(\delta_{ij}\phi - a_{ij})$ as the i, j entry, we see that $Av = 0$ for v the column matrix of generators (x_i).

Multiplying by the adjugate of A, we find that $\det(A)x_i = 0$ for each i and hence $\det(A)M = 0$. Employing the standard Laplace expansion for $\det(A)$ yields our monic polynomial, so we are done. □

Remark 1.14. We can actually conclude further by noting the form of our determinant that $a_i \in \mathfrak{a}^i$ where the a_i are as in (1.1).

Lemma 1.15 (Nakayama's Lemma)**.** *Let M be a finitely generated R-module and I an ideal of R such that $M = IM$. Then there exists $a \in I$ such that $(1 + a)M = 0$.*

Proof. Using the previous theorem, our ϕ is simply the identity mapping. Then, we find that $M + (a_1 + \ldots a_n)M = 0$. Setting $a_1 + \ldots a_n = a \in I$, we have that $(1 + a)M = 0$. □

Example 1.16. If $I \subset J(R)$ in the above, then $(1 + a)$ is a unit and we conclude further that $M = 0$.

Example 1.17. Suppose again that $I \subset J(R)$, but assume that $M = N + IM$ for some submodule N of M. Then, we see that $M/N = I(M/N)$, and employing the previous example, $M/N = 0$ so that $M = N$.

Consider the following construction: given an R-module M over a local ring $(R, \mathfrak{m}, k)$, take the quotient $M/\mathfrak{m}M \cong k \otimes M$. As a module over a field, this is actually a vector space. Choose a basis $\{\overline{x}_1, \ldots, \overline{x}_n\}$ of this vector space and consider the preimage $x_i \in M$ of each $\overline{x}_i$ with respect to the canonical projection. Then it is obvious that any proper subset of this set of generators cannot generate M (since else it would have to generate the vector space $k \otimes M$). Also, the set $X = \{x_1, \ldots, x_n\}$ generates all of M since for any $x \in M$ its image in $M/\mathfrak{m}M$ is in the span of our $\{\overline{x}_i\}$. Taking the preimage of this linear combination, we find that $x = r_1 x_1 + \cdots + r_n x_n$ for some $r_i \in R$. This motivates the following:

Definition 1.18. Let X be a generating set for an R-module M. If no proper subset of X generates M, then X is called a minimal basis.

In general, minimal bases need not contain the same number of elements. However, by our above construction, we have the following result for local rings:

Theorem 1.19. *Let $(R, \mathfrak{m}, k)$ be a local ring.*

(1) *For any basis of $M/\mathfrak{m}M$, its preimage will be a minimal basis of M.*
(2) *Conversely, every minimal basis is obtained in this manner.*
(3) *Given any two minimal bases $\{x_i\}$, $\{y_i\}$, $i = 1, \ldots n$, the matrix (a_{ij}) such that $y_i = \sum_i a_{ij} x_i$ is invertible over R.*

We conclude this section by defining two fundamental rings.

Definition 1.20. A ring is called *Noetherian* if every ascending chain of ideals eventually stabilizes. This is often called the ascending chain condition (ACC).

Definition 1.21. A ring is called *Artinian* if any descending chain of ideals eventually stabilizes. Similarly, this is often called the descending chain condition (DCC).

And we have the following

Theorem 1.22 (Akizuki)**.** *Every Artinian ring is Noetherian.*

1.2. **Localization of a Ring/Module.**

Definition 1.23. Let S be a multiplicative submonoid (hereafter referred to as a multiplicative subset) of a ring R. Then, the localization (or the ring of fractions) of R with respect to S is denoted either $S^{-1}R$ or R_S and is the set of equivalence classes of the form a/s with $a \in R$, $s \in S$. Two elements a/s and b/t are considered equivalent if $r(at - sb) = 0$ for some $r \in S$.

Addition is defined analogously to that of $\mathbb{Q}$: $\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}$, and multiplication as well: $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$.

Taking note of the definition of a prime ideal, we see that for $\mathfrak{p} \in \operatorname{Spec} R$ that if $a, b \notin \mathfrak{p}$, then $ab \notin \mathfrak{p}$. Hence, the complement of a prime ideal is a natural multiplicative subset, motivating our next definition.

Definition 1.24. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the localization of a ring R at $\mathfrak{p}$, denoted $R_{\mathfrak{p}}$, is the ring $S^{-1}R$ with $S = R \backslash \mathfrak{p}$.

It is easy to see that $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$, since any element r not belonging to $\mathfrak{p}$ has inverse $\frac{1}{r}$. Indeed, the localization

$R_{\mathfrak{p}}$ induces a natural one to one correspondence between the prime ideals contained in p and the prime ideals of $R_{\mathfrak{p}}$ by considering the natural inclusion $r \mapsto r/1$.

Localization of an R-module M is defined in a similar fashion, with our equivalence classes being of the form m/s, with $m \in M$ and $s \in S$. We consider $m/s = m'/s'$ if there exists $t \in S$ such that $t(ms' - m's) = 0$. Addition is defined as expected, and multiplication by elements of R is defined as $r\frac{m}{s} := \frac{rm}{s}$. In this way, it is clear that $S^{-1}M \cong S^{-1}R \otimes_R M$.

Definition 1.25. The support of a module M, denoted $\operatorname{Supp} M$, is defined as:

$$\operatorname{Supp} M := \{\mathfrak{p} \in \operatorname{Spec} R : \ M_{\mathfrak{p}} \neq 0\}$$

Localization tends to behave very well with respect to other ring/module operations. For example, we have that $S^{-1}R/S^{-1}I \cong S^{-1}(R/I)$ (on the right we are actually localizing at the image of S in R/I). Using this, we will use the notation $k(\mathfrak{p})$ to denote the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_p \cong (R/\mathfrak{p})_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$.

Localization of a ring also retains much of the structure of the original ring R, as shown in the following

Theorem 1.26. *Let R be a ring, S a multiplicative subset.*

(1) *All ideals of $S^{-1}R$ are of the form $S^{-1}I$, where I is an ideal of R.*
(2) *Every prime ideal of $S^{-1}A$ is of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal disjoint from S.*

Definition 1.27. An A-algebra B is a ring B equipped with a ring homomorphism $\phi : A \to B$.

If B is an A-algebra, then it has a natural A-module structure by defining the action of A as $a \cdot b = \phi(a)b$. How does localization of A affect B? Letting A_S denote the localization with respect to S, we want to make $B_{\phi(S)}$ into an A_S-algebra. However, the action is obvious in this case. Define

$$\frac{a}{s} \cdot \frac{b}{\phi(t)} := \frac{\phi(a)b}{\phi(st)}$$

(Of course, one would need to check that $\phi(S)$ remains multiplicative, but this is a trivial exercise.) This leads into the more general result:

Theorem 1.28. *Let A be a ring with $S \subset A$ a multiplicative set. Denote by $\psi : A \to A_S$ by the natural inclusion. If B is an A-algebra (with mapping ϕ) and there exists a homomorphism $g : B \to A_S$ such that $\psi = g \circ \phi$ and such that for every $b \in B$ there exists $s \in S$ such that $\phi(s) \cdot b \in \phi(A)$.*

Then, $A_S = B_{\phi(S)}$, and $\phi(S)$ consists precisely of the elements $b \in B$ such that $g(b)$ is a unit in A_S.

Using the above theorem, the most natural first situation is to consider a ring A with a multiplicative subset S, and suppose there exists some intermediate ring B such that $A \subset B \subset A_S$. Then, the mappings ϕ and g as above merely become inclusions, and we only need worry about when there exist $b \in B$ such that $bs = 0$ for some $s \in S$. We immediately deduce

Corollary 1.29. *Suppose $A \subset B \subset A_S$. If S contains no zero divisors, then A_S is also a ring of fractions for B. More precisely, $A_S = B_S$.*

And the following are also consequences of 1.28.

Corollary 1.30. *Let $\mathfrak{p} \in \operatorname{Spec} A$. Then, if B satisfies the conditions of 1.28, we have that $A_{\mathfrak{p}} = B_P$, where $P = \mathfrak{p}A_{\mathfrak{p}} \cap B$.*

Corollary 1.31. *Given two multiplicative sets S and T with $S \subset T$, we have that $(A_S)_{T'} = A_T$ (where T' denotes the image of T in A_S).*

We can now move on to some results which show how properties holding in a family of localizations of an R-module M give valuable information about M itself.

As a warm up, consider an element x such that the image of x in $M_{\mathfrak{m}}$ is 0 for every maximal ideal $\mathfrak{m}$. That means that for every maximal ideal $\mathfrak{m}$, there exists some $s \in \mathfrak{m}^c$ such that $sx = 0$. Thus $s \in \operatorname{Ann} x$, and since this holds for every $\mathfrak{m}$, we see that $\operatorname{Ann}(x)$ is not contained in any maximal ideal so that $\operatorname{Ann}(x) = R \implies x = 0$. We have proved

Theorem 1.32. *Let R be a ring, M an R-module with $x \in M$. If $x = 0$ in $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$, then $x = 0$.*

Using the above in combination with Nakayama's Lemma yields

Theorem 1.33. *Let R be a ring and M a finitely generated R-module. If $M \otimes_R k(\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$, then $M = 0$.*

And, more generally:

Theorem 1.34. *Let $f : A \to B$ be a ring homomorphism with M a finite B module. If $M \otimes_A k(\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Spec} A$, then $M = 0$.*