Degenerating the Jacobian

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Motivating Question

How to degenerate the Jacobian of a curve?

Review of the Jacobian

Recall that the *Jacobian* J_X of a genus g non-singular curve X is g-dimensional *abelian variety* that can be described in two different ways:

Albanese: J_X is the universal abelian variety into which X embeds;

Picard: J_X is the moduli space of degree 0 line bundles on X.

Families of Jacobians

Given a (proper, flat) family of curves $f\colon X \to B$ with non-singular fibers, the Jacobians of the fibers of f fit together to form a family $J \to B$.

What if some fibers are singular? It may be impossible to extend to a family of abelian varieties over B. This is the case for the family in Fig. 1.

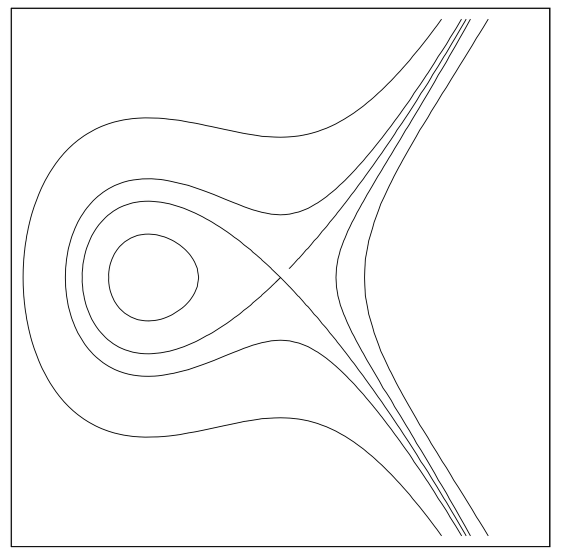

Figure 1: A family of plane cubics.

Refined Motivating Question

Given a (proper, flat) family of curves $f\colon X \to B$ with non-singular fibers over $b \in U \subset B$, how to extend the associated family of Jacobians $J_U \to U$ by adding degenerate fibers over the points $b \in B \setminus U$?

Two Approaches to Degeneration

Main Theorem (Thm. A of [Kass10])

Suppose that $f\colon X \to B$ is a projective family of geometrically reduced curves over a Dedekind scheme with the property that the local rings of X_S are factorial for every strict henselization $S \to B$ (e.g. X is regular). Let $J \to B$ be the locus of line bundles in one of the following families of compact moduli spaces of sheaves:

- the Esteves compactified Jacobian;
- the Simpson compactified Jacobian associated to a f-very ample line bundle L with the property that every L-slope semi-stable, pure, rank 1 sheaf of degree 0 is slope stable.

Then J is the Néron model of its generic fiber.

In other words, this theorem relates two different approaches to degenerating the Jacobian.

Approach 1: the Néron model

Viewing the Jacobian J_X as the *Albanese variety*, it is natural to try to extend $J_U \to U$ by adding degenerate group varieties. This leads to the Néron model.

Set-Up: Let A_η be an abelian variety over the generic point η of a Dedekind scheme B.

Definition: The **Néron model** $\mathrm{N}(A_\eta)$ is a smooth B-model of A_η that satisfies the Néron mapping property: given T a smooth B-scheme, every morphism $T_\eta \to A_\eta$ extends uniquely to $T \to \mathrm{N}(A_\eta)$.

Existence: Néron proved that $\mathrm{N}(A_\eta)$ exists [Nér64].

Taking $A_\eta = J_{X_\eta}$, Néron's work provides an extension of the Jacobian to a family over B.

Example 1: For Fig. 1, the special fiber of $\mathrm{N}(A_\eta)$ is the multiplicative group $\mathbb{G}_m$.

Example 2: For Fig. 2, the special fiber is the disconnected group $\mathbb{Z}/2 \times \mathbb{G}_m$.

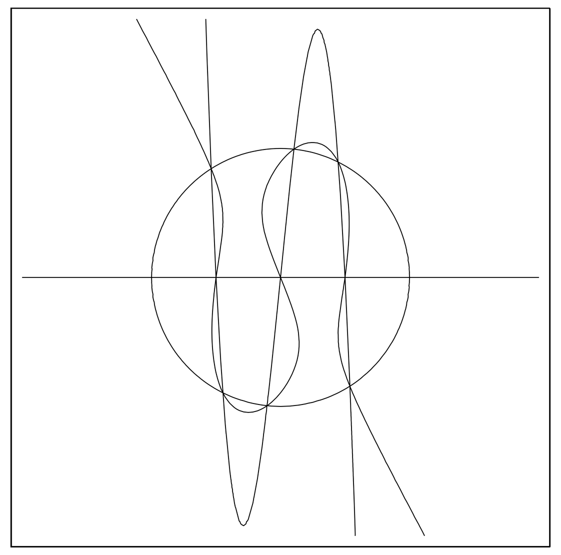

Figure 2: A second family of plane cubics.

Approach 2: Stable sheaves

The description of J_X as the *Picard variety* suggests that we should extend $J_U \to U$ by adding degenerate moduli spaces of sheaves.

Problem: For X_b singular, there is *no* algebraic moduli space parameterizing degree 0 line bundles and their degenerations. (See Fig. 3.)

Solution: Impose a numerical condition on the multi-degree of a line bundle. Lots of papers on this: [AK80], [Cap94], [D'S79], [Est01], [Ish78], [Jar00], [OS79], [Pan96], [Sim94]....

Stability: If (X_b, L_b) is a polarized curve, then a pure sheaf I is **slope semi-stable** if the slope inequality $\mu(J) \leq \mu(I)$ holds for all $J \subset I$.

Existence: The fine moduli space of stable sheaves with fixed Hilbert polynomial exists by [Sim94]. The **Simpson compactified Jacobian** is a connected component. Imposing a different numerical condition, we get the **Esteves compactified Jacobian**.

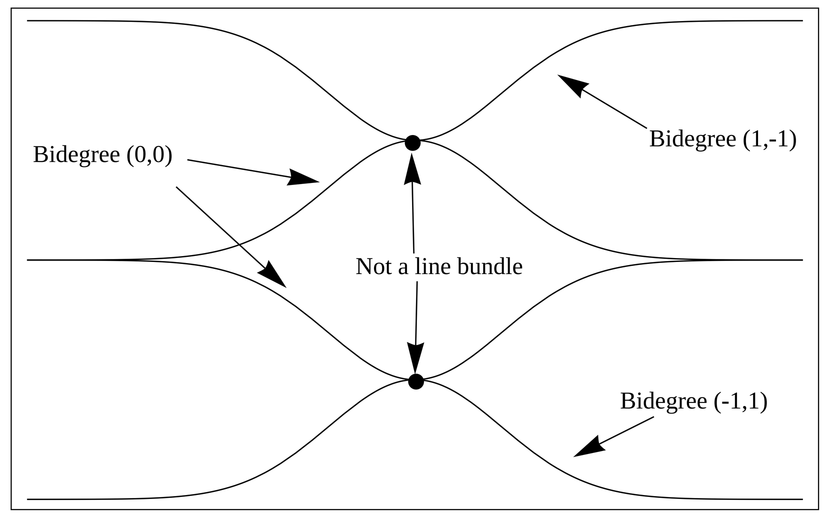

Figure 3: Pure sheaves on the singular curve in Fig. 2.

Remarks on the theorem

Hypotheses: When the fibers X_b are nodal, suitable polarizations L exist [MV10].

Generalization: Theorem 4.8 of [Kass10] applies when fibers are non-reduced.

Non-reduced: For rational ribbons, Chen and I [ChKa11] proved a space exists iff genus is even (answering a question of Green–Eisenbud).

Related Results: See [Cap10], [OS79], [MV10]. Proof of Theorem A is different (no combinatorics).

Application: The Simpson Jacobian compactifies the Néron model. In [Kass09], I used this fact to partially answer a question of Lang [Lan83].

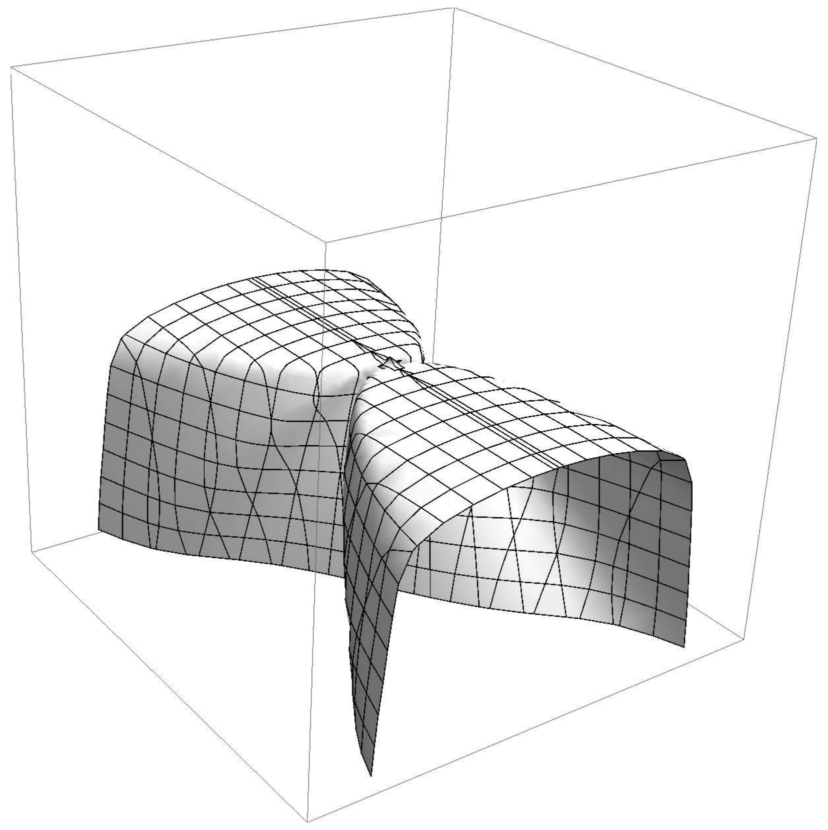

Figure 4: The Simpson Jacobian of a genus 2 curve.

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