Lecture Notes on Compactified Jacobians

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Abstract

In these notes, we discuss the some aspects of the theory of compactified Jacobians. The focus is on explaining the basic results in a modest level of generality and giving some concrete examples. These notes are based on a talk that the author gave at a student seminar at Harvard in Fall 2007.

Contents

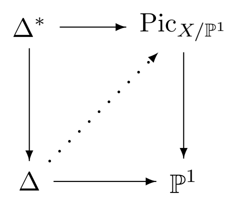
1 Motivation

A fundamental problem in algebraic geometry is "how to take limits of line bundles?" A simple, but illustrative example can be constructed from the geometry of plane cubics. Let X_0 be a plane cubic curve in $\mathbb{P}^2$ with a node at the origin $p_0 = [0, 0, 1]$ and X_∞ be a general cubic curve that passes through p_0. Consider the pencil $\{X_t\}$ spanned by these curves. For $t \neq 0$, we can define a line bundle $\mathscr{L}_t$ by setting $\mathscr{L}_t = \mathscr{O}_{X_t}(-p_0)$ (the ideal sheaf of p_0). It is natural to ask "what is the limit"

$$\lim_{t \to 0} \mathscr{L}_t = ?$$

This limit wants to be the ideal sheaf of p_0 in X_0, but this sheaf is not a line bundle.

One can restate this question more formally. Let $X \to \mathbb{P}^1_t$ be the pencil in question. There is a scheme $\mathrm{Pic}_{X/\mathbb{P}^1} \to \mathbb{P}^1$, called the **relative Picard scheme** that parametrizes families of line bundles on the given pencil of curves. Let $\Delta = \mathrm{Spec}(k[[t]])$ be a formal disc around the origin of $\mathbb{P}^1$ and $\Delta^* = \mathrm{Spec}(\mathrm{Frac}(k[[t]]))$ the formal punctured disc. The line bundles $\mathscr{L}_t, t \neq 0$, fit together to form a family of line bundles and hence induce a morphism $\Delta^* \to \mathrm{Pic}_{X/\mathbb{P}^1}$. This morphism fits into the diagram below:



Because the ideal sheaf I_{p_0} of X_0 is not a line bundle, it follows that there is no extension of $\Delta^* \to \mathrm{Pic}_{X/\mathbb{P}^1}$ to a morphism $\Delta \to \mathrm{Pic}_{X/\mathbb{P}^1}$. This shows that $\mathrm{Pic}_{X/\mathbb{P}^1}$ does not satisfy the valuative criteria of properness. The question "how to take limits of line bundles?" can be restated as "how to compactify the relative Picard scheme?"

There is no canonical answer to this question. A wide range of different compactifications have been constructed. Each compactification has its own advantages and disadvantages. The "correct" compactification to consider depends on the context. In these notes, we will discuss a compactification of the Picard scheme using torsion-free sheaves. The idea behind this construction is attributed to Alan Mayer and David Mumford.

Inspired by some work of Igusa, they proposed a construction of a compactification of the Jacobian of a curve using rank 1, torsion-free sheaves at a conference in 1964. In his thesis [5], D'Souza gave a construction of this compactification for the relative Jacobian of a family of curves over a Henselian (Noetherian) local ring with separably closed residue field. His proof uses methods from geometric invariant theory. In [2], Allen Altman and Steven Kleiman constructed the relative compactified Picard scheme in greater generality. Their proof was modeled on Grothendieck's construction of the relative Picard scheme and will be discussed later in these notes. Much of the material in these notes is derived from Altman and Kleiman's article [2].

The Altman-D'Souza-Kleiman theory has the following features:

Advantages:

- The compactification is a fine moduli space for a natural class of algebro-geometric objects.
- The Jacobian group variety acts on the compactification.

- There is an Abel map generalizing the classical Abel map from a smooth curve to its Jacobian.

Disadvantages:

- The theory only works for integral curves.
- For integral curves with sufficiently bad singularities, the compactification is "too big" (i.e. there are boundary components whose dimension is larger than the dimension of the Picard scheme).

In the example with plane cubics, the limit of the line bundles $\mathscr{L}_t$ in the Altman-D'Souza-Kleiman compactification is the ideal sheaf I_{p_0} of the point p_0 on the curve X_0. In general, the Picard scheme is compactified by allowing line bundles to degenerate to rank 1, torsion-free sheaves.

2 Basic Definitions and Theorems

2.1 The Picard Functor

The theory of the compactified Jacobian works for a fairly arbitrary family of integral curves. For the remainder of this section, let $f : X \to S$ be locally projective, finitely presented, and flat with 1-dimensional integral geometric fibers. We shall occasionally use the term **"family of integral curves"** as a shorthand to refer to such morphism.

There is some disagreement in the literature on what it means for a morphism to be locally projective. In these notes, we follow the conventions of [8]. The relevant sections are 5.3 and 5.5.

If $f : X \to S$ is a morphism of schemes, then we say that $f : X \to S$ is **projective** if X is isomorphic to a closed subscheme of $\mathbb{P}(\mathscr{E})$ over S for some quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$ that is of finite type. In general we say that $f : X \to S$ is **quasi-projective** if $f : X \to S$ is of finite type and admits a relatively amply line bundle. When S is quasi-compact and quasi-separated, this is equivalent to saying that X is isomorphism to a locally closed subscheme of $\mathbb{P}(\mathscr{E})$ over S for some quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$ that is of finite type. The condition of being quasi-projective is not local on the base S. We say that $f : X \to S$ is **locally quasi-projective** (resp. **locally projective**) if there is an open cover of S such the restriction of f is any element of the open cover is quasi-projective (resp. projective).

For a family $f : X \to S$ of integral curves, the (arithmetic) genus of a geometric fiber $X_{\bar{s}}$ defines a locally constant function of the base S. We will *assume that the genus is in fact constant and equal to g*. Furthermore, to simplify various technical issues, let us suppose that we are also given a **section** $\sigma : S \to X$ contained in the smooth locus of f. Let us now review the definition of the relative Picard functor.

The **relative Picard functor**, written $\mathrm{Pic}_{X/S}$ or $P_{X/S}$, is defined to be the functor from S-schemes to sets that is given by the rule:

$$\mathrm{Pic}_{X/S}(T) = \{(\mathscr{L}, i) : \mathscr{L} \text{ is a line bundle on } X_T, i \text{ is an isom. } \sigma^*(\mathscr{L}) \cong \mathscr{O}_T\}/\cong$$

Here $\cong$ indicates that we are taking pairs up to isomorphism. An isomorphism from one pair $(\mathscr{L}, i)$ to another pair $(\mathscr{M}, j)$ is an isomorphism $\mathscr{L} \to \mathscr{M}$ that respects the identifications i and j. We will occasionally be sloppy and refer to an element of $\mathrm{Pic}_{X/S}(T)$ as a line bundle or a sheaf even though strictly speaking such an element is a line bundle with some additional data.

A given pair $(\mathscr{L}, i)$ has no non-trivial automorphisms, so that it follows from descent for quasi-coherent sheaves that $\mathrm{Pic}_{X/S}$ is a sheaf (on, say, the big étale site). This would not be true if we only considered line bundles up to isomorphism. It is for this reason that we have made the simplifying assumption that we are given a section σ that is contained in the smooth locus.

The degree of a line bundle is locally constant in flat families, so we can obtain a component of $\mathrm{Pic}_{X/S}$ by considering families of fixed degree. The subfunctor of $\mathrm{Pic}_{X/S}$ that parametrizes families of degree n line bundles is denoted by $\mathrm{Pic}^n_{X/S}$ or $P^n_{X/S}$. The subfunctor $\mathrm{Pic}^0_{X/S}$ is also known as the **relative Jacobian** and we will also be written as $J_{X/S}$. Because we have assumed that $f : X \to S$ admits a section, the functors $\mathrm{Pic}^n_{X/S}$ are all isomorphic.

There are several different theorems about the representability of the Picard functor. For our purposes, the following result more than suffices:

Theorem 1. *Suppose that $f : X \to S$ is locally projective, finitely presented, and flat with 1-dimensional integral geometric fibers. Then $\mathrm{Pic}_{X/S} \to S$ is representable by a S-scheme that is smooth of relative dimension g. Furthermore, each subscheme $\mathrm{Pic}^n_{X/S} \to S$ is locally quasi-projective over S.*

Proof. This theorm is originally due to Grothendieck [9]. One recent exposition of the proof is [10].

The proof of the existence of the compactified Picard scheme follows along the same lines as this proof, so it is worth recalling the general flavor of the argument. Let us suppose for simplicity that S is the spectrum of an algebraically closed field, so that we are dealing with a single curve. Furthermore, let's assume that X is in fact a smooth curve. To prove the representability of $\mathrm{Pic}_{X/S}$, one can show that it is enough to prove the representability of $\mathrm{Pic}^n_{X/S}$ for some n. For any n, there is a natural map $\mathrm{Sym}^n(X) \to \mathrm{Pic}^n_{X/S}$ given by $\{p_1, \ldots, p_n\} \mapsto \mathscr{O}_X(p_1 + \ldots + p_n)$. For n sufficiently large, this map realizes the symmetric power of X as a projective bundle over the functor $\mathrm{Pic}^n_{X/S}$. Finally, one shows that it is possible to take the quotient of the symmetric power of X by the equivalence relation defined by identifying points lying in the same fiber of $\mathrm{Sym}^n(X) \to \mathrm{Pic}_{X/S}$. □

Example: In the example of the pencil of plane cubics $f : X \to S$ from the introduction, the relative Jacobian $J_{X/S}$ is (non-canonically) isomorphic to the smooth locus $X^0 \to S$ of $f : X \to S$. The general (geometric) fiber of this family is an elliptic curve. There are a finite number of special fibers (corresponding to the nodal curves) for which the (geometric) fiber is a group variety isomorphic to $\mathbb{G}_m$.

We compactify $\mathrm{Pic}_{X/S}$ by enlarging the functor of points. In order to define the compactified Picard functor, it is necessary to make some further definitions.

2.2 Some Basic Facts about Torsion-free Sheaves

Suppose for the moment that S is the spectrum of an algebraically closed field, so that we are dealing with a single integral curve rather than a family of such curves. To emphasize that we are working with a single curve, let us write $\bar{s}$ for S and $X_{\bar{s}}$ for X. We say that a coherent sheaf on $X_{\bar{s}}$ is **rank 1, torsion-free** if it satisfies the following conditions:

1. The sheaf I satisfies Serre's condition $S1$ (no associated prime is embedded).
2. The sheaf I is generically isomorphic to $\mathscr{O}_{X_{\bar{s}}}$.

Because we are dealing only with integral curves, condition 1 is equivalent to the condition that no non-zero section of $\mathscr{O}_{X_{\bar{s}}}$ kills a non-zero section of I (i.e. I is torsion-free in the "usual sense"). Condition 1 is useful as it is more geometric and it generalizes more readily to the non-integral case. We should also remark that if $X_{\bar{s}}$ is regular, then every rank 1, torsion-free sheaf is a line bundle. This is, for example, a consequence of the classification of finite modules over a discrete valuation ring.

Examples: Basic examples of rank 1, torsion-free sheaves are:

- Any line bundle.
- Any non-zero ideal sheaf.

Some examples of sheaves that fail to be rank 1, torsion-free are:

- Any vector bundle of rank greater than 1. Such a sheaf is not generically isomorphic to $\mathscr{O}_{X_{\bar{s}}}$.
- Any sheaf that is supported on a proper subset of X. Such a sheaf is also not generically isomorphic to $\mathscr{O}_{X_{\bar{s}}}$.
- The direct sum of a line bundle and a skyscraper sheaf or, more generally, any extension of a line bundle by a skyscraper sheaf. Such a sheaf has an associated prime that is embedded.

Now suppose that S is again arbitrary. Let us define what is meant by a family of torsion-free, rank 1 sheaves. Suppose that I is a locally finitely presented $\mathscr{O}_X$-sheaf on X. We say that I is a **S-relatively rank 1, torsion-free sheaf** or a family of rank 1, torsion-free sheaves parametrized by S if the following two conditions are satisfied:

1. I is S-flat.
2. The restriction of I to any geometric fiber $I|_{X_{\bar{s}}}$ is a torsion-free, rank 1 sheaf in the previously defined sense.

With these definitions out of the way, we now define the **relative compactified Picard scheme** by enlarging the functor of points of $P_{X/S}$. We define relative compactified Picard functor, written $\bar{\text{Pic}}_{X/S}$ or $\bar{P}_{X/S}$, by the rule

$$\bar{\text{Pic}}_{X/S}(T) = \{(I, i) : I \text{ is a rel. tor.-free, rnk 1 sheaf on } X_T, i : \sigma^*(I) \cong \mathscr{O}_T\}/\cong$$

Here $\cong$ again means that we are taking pairs $(\mathscr{L}, i)$ up to isomorphism.

The locus of sheaves with fixed numerical invariants is a component of $\bar{\text{Pic}}_{X/S}$ and so the compactified Picard functor breaks up into a countable number of components. Define the **degree** of a single rank 1, torsion-free sheaf I by the formula:

$$\deg(I) = \chi(I) - \chi(\mathscr{O}_{X_{\bar{s}}})$$

If I is a S-relatively rank 1, torsion-free sheaf on X, then the fiber-wise degree of I defines a locally constant function of the base S. Set $\bar{\text{Pic}}^n_{X/S}$ equal to be the subfunctor of $\bar{\text{Pic}}_{X/S}$ that parametrizes families of degree n rank 1, torsion-free sheaves.

The functor $\bar{\text{Pic}}^0_{X/S}$ is also known as the **relative compactified Jacobian** and is denoted $\bar{J}_{X/S}$. To ease notation, we will often write $\bar{P}_{X/S}$ or $\bar{P}$ for $\bar{\text{Pic}}_{X/S}$ and $\bar{P}^n_{X/S}$ or $\bar{P}^n$ for $\bar{\text{Pic}}^n_{X/S}$.

Remark: There is perhaps one unexpected difference between the definition of the relative Picard functor and the relative compactified Picard functor. We defined the relative Picard functor so that a morphism from a S-scheme T to $P_{X/S}$ corresponds to a line bundle on X_T together with a local trivialization. By contrast, the relative compactified Picard functor was NOT defined by requiring that the morphisms from T to $\bar{P}_{X/S}$ correspond to rank 1, torsion-free sheaves on X_T together with a local trivialization. Instead, these morphisms correspond to T-flat sheaves on X_T that are fiber-wise rank 1, torsion-free. This is not the same as requiring that I is globally rank 1, torsion-free. The definition that we have chosen is the definition that most accurately captures the intuition of a "family of rank 1, torsion-free sheaves".

In analogy with this definition of "family of rank 1, torsion-free sheaves", one might be tempted to redefine the relative Picard scheme $P_{X/S}$ so that morphisms from T to $P_{X/S}$ corresponds to T-flat sheaves on X_T that are fiber-wise locally free of rank 1. One can show that such sheaves are precisely the line bundles on X_T, so this alternative definition leads to an isomorphic functor. The poof of this fact is an application of standard theorems on cohomology and base change.

2.3 The Existence Theorem

The basic existence theorem is:

Theorem 2. *Suppose that $f : X \to S$ is locally projective, finitely presented, and flat with 1-dimensional integral geometric fibers. Then the functor $\bar{Pic}_{X/S}$ is representable by a S-scheme. Furthermore, $\bar{Pic}^n_{X/S}$ is finitely presented and locally projective over S.*

Proof. This is a combination of theorems 8.1 and 8.5 from [2]. Under more restrictive hypotheses, this theorem is proven in [5].

The proof in [2] follows along the same lines as the proof that the Picard scheme of a family of integral curves exists that was sketched earlier. One major difficulty in generalizing the construction of the relative Picard scheme lies in developing a theory of the Abel map in sufficient generality. We refer the reader to the literature for details. □

Remark:

- Observe that it follows from this theorem that $\bar{P}_{X/S} \to S$ is proper and so $\bar{P}_{X/S}$ is a genuine compactification of $P_{X/S}$! The tensor product of a line bundle and a rank 1, torsion-free sheaf is again a rank 1, torsion-free sheaf so the scheme $\bar{P}_{X/S}$ admits a natural action by $P_{X/S}$. In the next section, we will discuss the Abel map from the Hilbert scheme to the relative compactified Picard scheme. These three properties are the three fundamental properties of the relative compactified Picard scheme that were stated in the introduction.

- If one assumes stronger projectivity conditions for $f : X \to S$, then one can conclude stronger projectivity conditions for $\bar{P}^n_{X/S} \to S$. We refer the interested reader to theorem 8.5 in [2] for the details.

- Notice that the theorem does not say that $\bar{P}_{X/S} \to S$ is smooth of relative dimension g. In fact, the morphism $\bar{P}_{X/S} \to S$ is never smooth except in trivial cases. Furthermore, there are example where the geometric fibers of $\bar{P}_{X/S} \to S$ have dimension greater than g although this is a somewhat pathological phenomenon. We shall see illustrations of both of these properties in the examples at the end of these notes.

- When S is the spectrum of an algebraically closed field, Eduardo Esteves has given a third proof of this theorem. In [6], he constructs the compactified Jacobian by using what he calls "theta sections" of a vector bundle. His approach is particularly useful in studying the projective geometry of the compactified Jacobian.

2.4 The Abel Map

The Abel map is a fundamental tool in the study of the compactified Jacobian. The theory of the Abel map is somewhat less technical for Gorenstein curves. For the purposes of exposition, we will focus on the Gorenstein case and direct the reader interested in the non-Gorenstein case to [2].

For this subsection, we will now assume that $f : X \to S$ satisfies all of the conditions listed at the beginning of section 2.1 and has the additional property that *the geometric fibers $X_{\bar{s}} \to \bar{s}$ are Gorenstein.* The degree d component of the **Abel map** $A^d : \mathrm{Hilb}^d_{X/S} \to \bar{\mathrm{Pic}}^{-d}_{X/S}$ is defined in terms of a map on the associated functors of points. Morally, the Abel map is defined to be the map that takes a degree d closed subscheme Z to the ideal sheaf of Z, considered as a coherent sheaf. This definition does not quite work because the ideal sheaf I_Z may not be trivial when restricted to the section σ. Instead, we define A^d as follows. Suppose that T is an arbitrary S-scheme. Then a T-valued point of $\mathrm{Hilb}^d_{X/S}$ is a T-flat closed subscheme of X_T with fiber-wise degree d. The **image of this point under the Abel map** is defined to be the T-valued point of $\bar{P}^{-d}$ corresponding to the following sheaf:

$$(I_Z \otimes f^*(\sigma^*(I_Z)^\vee), i_{\mathrm{can}})$$

Here i_{can} is the canonical trivialization of the restriction of $I_Z \otimes f^*(\sigma^*(I_Z)^\vee)$ to σ (coming from the identity $f \circ \sigma = \mathrm{id}$).

Observe that this notation is inconsistent with the notation traditionally used for the Abel map. The Abel map of a smooth curve is usually defined so that it sends a degree d divisor to a degree d line bundle. We have chosen our notation to agree with the conventions used in [2]. One advantage of this convention is that it avoids dualizing the ideal sheaf. In the non-Gorenstein case, the ideal sheaf of a closed subscheme of a curve may fail to be reflexive so the operation of dualization is somewhat ill-behaved.

Before stating the main result on the Abel map, it is convenient to introduce some terminology. For the purposes of defining terminology, we need to momentarily assume that S is the spectrum of an algebraically closed field so that we are dealing with a single integral Gorenstein curve. We will again write $\bar{s}$ for S and $X_{\bar{s}}$ for X when we are working with a single curve. If I is a rank 1, torsion-free sheaf on the curve $X_{\bar{s}}$, then the **index of specialty** of I is defined to be the dimension of the space

$$\mathrm{Ext}^1(I, \mathscr{O}_{X_{\bar{s}}})$$

If I is actually a line bundle, then this cohomology group is isomorphic to $H^1(X_{\bar{s}}, I^\vee)$. Taking into account that our conventions are "dual" to the standard ones, this confirms that our definition of the index of speciality generalizes the usual notion for line bundles. We say that I is **special** if the index of specialty is non-zero and that I is **non-special** otherwise.

With these preliminaries out of the way, we can state the basic properties of the Abel map:

Theorem 3. *Let $f : X \to S$ be flat, finitely presented, and locally projective with geometric fibers equal to integral Gorenstein curves of genus g. Fix an integer d and consider the Abel map*

$$A^d : \mathit{Hilb}^d_{X/S} \to \bar{P}^{-d}_{X/S}$$

Then the following properties hold:

1. *The map A^d is surjective if and only if $d \geq g$.*
2. *The locus of non-special points of $\bar{Pic}^d_{X/S}$ is open and the map A^d is smooth of relative dimension $d-g$ over this open subscheme.*
3. *The map A^d is smooth of relative dimension $d-g$ if and only if $d \geq 2g-1$.*

Proof. This is a combination of theorem 8.4 on page 103 of [2] and theorem 8.6 on page 106 of the same paper. The idea is to relate the fibers of the Abel map over a point I to a projective space related to the cohomology of I. □

Remark:

- Regardless of whether or not the geometric fibrs of f are Gorenstein, there is always a natural map $A^d : \text{Hilb}^d_{X/S} \to \bar{P}^{-d}$. However, if the family $f : X \to S$ contains non-Gorenstein curves, then one can show that this map is not a smooth fibration.
- It is possible to adapt this theorem to work for families containing non-Gorenstein curves by taking the source of the Abel map to be the Quot scheme $\text{Quot}^d_{\omega/X/S}$, rather than the Hilbert scheme. Here ω is the relative dualizing sheaf.

3 Examples

In this section, we explicitly describe the geometry of some compactified Jacobians and then say some words about what is known in general. For the remainder of this note, we will focus primarily on a single curve so that S is the spectrum of an algebraically closed field k. As before, we will write $\bar{s}$ for S and $X_{\bar{s}}$ for X when the base S is the spectrum of an algebraically closed field.

3.1 Smooth Curves

On a smooth curve every rank 1, torsion-free sheaf is a line bundle. In particular if $X_{\bar{s}}/\bar{s}$ is a smooth curve, then we have that $P_{X_{\bar{s}}/\bar{s}} = \bar{P}_{X_{\bar{s}}/\bar{s}}$. This is what one expects since the Jacobian of a smooth curve is compact and so there is no need to compactify.

3.2 Genus 1 Curves

Now suppose that $X_{\bar{s}}/\bar{s}$ is a singular integral curve of genus 1, so that the curve is either a nodal plane cubic or a cuspidal plane cubic. In this case, the degree -1 component $\bar{P}^{-1}_{X_{\bar{s}}/\bar{s}}$ of the relative compactified Picard scheme is canonically isomorphic to $X_{\bar{s}}$ and hence the compactified Jacobian $J_{X_{\bar{s}}/\bar{s}}$ is (non-canonically) isomorphic to $X_{\bar{s}}$. This fact remains true even if the base is allowed to be arbitrary. In particular, if $f : X \to S$ is the family of plane cubics from the introduction then relative compactified Jacobian $\bar{J}_{X/S}$ is isomorphic to X.

The proof goes as follows. A family of integral curves always embeds into its relative compactified Jacobian provided the genus g is greater than zero:

Lemma 1. *Suppose that $f : X \to S$ is locally projective, finitely presented, and flat with 1-dimensional integral geometric fibers of genus $g > 0$. Then the Abel map $p \mapsto (I_p \otimes f^*(\sigma^*(I_p)^\vee), i_{can})$ is a closed embedding $X \to \bar{P}^{-1}_{X/S}$.*

Proof. This is theorem 8.8 on page 108 of [2]. The morphism $X \to \bar{P}^{-1}_{X/S}$ is proper and finitely presented, so it enough to prove that any scheme-theoretic fiber of a geometric point of $\bar{P}^{-1}_{X/S}$ is either empty or a single reduced point. This face is proven using the interpretation of the fibers of the Abel map in terms of the projectivization of certain cohomology groups. □

Now in the case that the genus is 1, one can additionally conclude from theorem 3 that the map $X \to \bar{P}^{-1}_{X/S}$ is also smooth of relative dimension 0. It follows that the map is an isomorphism.

3.3 Genus 2 Curves with a Node

Suppose that $X_{\bar{s}}/\bar{s}$ is a genus 2 curve with a single node. Let $X'_{\bar{s}}/\bar{s}$ be the normalization and $g : X'_{\bar{s}} \to X_{\bar{s}}$ the normalization map. Say that q is the nodal point of $X_{\bar{s}}$ and that p_1 and p_2 are the two points of $X'_{\bar{s}}$ that lie above q. The structure of the (non-compactified) Jacobian of $X_{\bar{s}}$ is easy to describe. The Jacobian $J_{X'_{\bar{s}}/\bar{s}}$ is an elliptic curve. There is a natural map $h : J_{X_{\bar{s}}/\bar{s}} \to J_{X'_{\bar{s}}/\bar{s}}$. In terms of moduli of line bundles, this map is defined by pullback via g. This map realizes $J_{X_{\bar{s}}/\bar{s}}$ as a $\mathbb{G}_m$-bundle over $J_{X'_{\bar{s}}/\bar{s}}$:

$$0 \to \mathbb{G}_m \to J_{X_{\bar{s}}/\bar{s}} \to J_{X'_{\bar{s}}/\bar{s}} \to 0$$

The compactified Jacobian $\bar{J}_{X_{\bar{s}}/\bar{s}}$ can be explicitly constructed in terms of a certain completion of $J_{X_{\bar{s}}/\bar{s}}$ to a $\mathbb{P}^1$-bundle over $J_{X'_{\bar{s}}/\bar{s}}$. We will just briefly sketch the construction and omit all proofs. The details are carefully worked out in [3]. See [7] for later developments.

As a $\mathbb{G}_m$-bundle over $J_{X'_{\bar{s}}/\bar{s}}$, the scheme $J_{X_{\bar{s}}/\bar{s}}$ can be completed to a $\mathbb{P}^1$-bundle over $J_{X'_{\bar{s}}/\bar{s}}$ by adding in two sections: a section σ_1 "at infinity" and a section σ_2 "at zero." After unwinding the definition of the $\mathbb{G}_m$-bundle structure, one can check that the correct completion to take is the projectivization of the sheaf $(1 \times h)_*(\wp)|_{q \times J_{X'_{\bar{s}}/\bar{s}}}$ on $J_{X'_{\bar{s}}}$. Here $\wp$ is the Poincare bundle on $X \times J_{X_{\bar{s}}/\bar{s}}$. We will let Pres_g denote this projective bundle. This notation is chosen to agree with the notation used in [3]. In that paper, the projective bundles that are constructed in this manner are called presentation schemes.

There is a natural map $\mathrm{Pres}_g \to \bar{J}_{X_{\bar{s}}/\bar{s}}$ that realizes $\bar{J}_{X_{\bar{s}}/\bar{s}}$ as a quotient of Pres_g. This map identifies the section σ_1 with the section σ_2 and is an isomorphism away from these sections. However, the two sections do not get identified fiber-wise! The line bundle $\mathscr{O}_{X'_{\bar{s}}}(q_1 - q_2)$ represents an element t of $J_{X'_{\bar{s}}/\bar{s}}(\bar{s})$. The two sections σ_1 and σ_2 are identified in such a way that if $g \in J_{X'_{\bar{s}}/\bar{s}}(\bar{s})$ then the point $\sigma_1(g)$ is identified with the point $\sigma_2(g + t)$. Here addition is the group law on $J_{X'/k}$. As an aside, it follows that the natural map $J_{X_{\bar{s}}/\bar{s}} \to J_{X'_{\bar{s}}/\bar{s}}$ does NOT extend to a map $\bar{J}_{X_{\bar{s}}/\bar{s}} \to J_{X'_{\bar{s}}/\bar{s}}$ of the compactified Jacobians.

The boundary $\partial(\bar{J}_{X_{\bar{s}}/\bar{s}})$ of the compactified Jacobian is equal to the image of the two sections σ_1 and σ_2 under the natural map, so the boundary is irreducible and 1-dimensional. The compactified Jacobian is singular along the boundary. At a point on the boundary, the compactified Jacobian locally looks like two smooth surfaces meeting transversely.

Given that there is a 1-dimensional irreducible scheme parametrizing rank 1, torsion-free sheaves on $X_{\bar{s}}$ that are not locally free, it is natural to ask how these sheaves can be described "explicitly". If $\mathscr{L}$ is a degree -1 line bundle on $X'_{\bar{s}}$, then $g_*(\mathscr{L})$ is a torsion-free rank 1 sheaf on $X_{\bar{s}}$ that fails to be locally free at the node. By a dimension count, it follows that every such sheaf must be of this form. One can further develop this line of thinking to construct an isomorphism $\partial(J_{X_{\bar{s}}/\bar{s}}) \cong P^{-1}_{X'_{\bar{s}/\bar{s}}} (\cong X'_{\bar{s}})$.

We can even describe the action of $J_{X_{\bar{s}}/\bar{s}}$ on the boundary. The subgroup $\mathbb{G}_m$ acts trivially on the boundary and the action of $J_{X_{\bar{s}}/\bar{s}}$ on the boundary factors through the quotient $J_{X'_{\bar{s}}/\bar{s}}$. Under the identification $\partial(J_{X_{\bar{s}}/\bar{s}}) \cong \bar{P}^{-1}_{X'_{\bar{s}/\bar{s}}}$, the action of $J_{X'_{\bar{s}}/\bar{s}}$ on $\partial(J_{X_{\bar{s}}/\bar{s}})$ is identified with the natural action on $\bar{P}^{-1}_{X'_{\bar{s}/\bar{s}}}$

3.4 Genus 2 Curves with a Cusp

Let $X_{\bar{s}}/\bar{s}$ be a genus 2 curve with a cusp. There is a description of the compactified Jacobian of this curve that is analogous to the description of the compactified Jacobian of a genus 2 curve with a node. The (non-compactified) Jacobian of $X_{\bar{s}}$ is an extension of an elliptic curve by the additive group $\mathbb{G}_a$. To be more precise, let $X'_{\bar{s}}/\bar{s}$ denote normalization and $g : X'_{\bar{s}} \to X_{\bar{s}}$ the normalization map. There is a natural surjection $J_{X_{\bar{s}}/\bar{s}} \to J_{X'_{\bar{s}}/\bar{s}}$ with kernel equal to $\mathbb{G}_a$.

To explicitly construct the compactified Jacobian, we first complete $J_{X_{\bar{s}}/\bar{s}}$ to a $\mathbb{P}^1$-bundle over $J_{X'_{\bar{s}}/\bar{s}}$ that we denote by Pres_g. In the case of a nodal genus 2 curve, the projective bundle was constructed by adding in a "section at infinity" and a "section at zero." In the cuspidal case, the projective bundle is obtained by adding in a length 2 non-reduced subscheme "at infinity" that is supported along a section. More rigorously, this projective bundle can be constructed in terms of the Poincare sheaf as in the nodal case. There is a natural map $\mathrm{Pres}_g \to \bar{J}_{X_{\bar{s}}/\bar{s}}$ that is an isomorphism "away from infinity" and collapses the length 2 non-reduced subscheme at infinity to its underlying reduced subscheme.

The description of the boundary $\bar{J}_{X_{\bar{s}}/\bar{s}}$ follows along the same lines as the case of a genus 2 curve with a node. The boundary is isomorphic to $\bar{P}^{-1}_{X'_{\bar{s}/\bar{s}}} \cong X'_{\bar{s}}$ and, in particular, is 1-dimensional and irreducible. The action of the additive subgroup of $J_{X_{\bar{s}}/\bar{s}}$ on the boundary is trivial, so the action factors through the quotient $J_{X'_{\bar{s}}/\bar{s}}$. This action can be identified with the natural action of $\bar{J}_{X'_{\bar{s}/\bar{s}}}$ on $\bar{P}^{-1}_{X'_{\bar{s}/\bar{s}}}$.

3.5 The General Picture

The examples that we have discussed so far in these notes are somewhat misleading. If $X_{\bar{s}}$ is a singular integral curve, then it is always possible to give an explicit description of the Jacobian of $X_{\bar{s}}$ in terms of the Jacobian of the normalization of $X_{\bar{s}}$ and the structure of the singularities. Such a description can be found in many textbooks such as section 9.2 of [4] or chapter 3 of [13]. It is NOT always possible to given such a description of the compactified Jacobian. When $X_{\bar{s}}$ has only double points (e.g. nodes, cusps, tacnodes, ect), the theory of presentation schemes provides a description of the geometry of the compactified Jacobian that generalizes our discussion of the case when $X_{\bar{s}}$ is of genus 2 and has only a node or a cusp as a singularity.

By contrast, the geometry of the compactified Jacobian seems to be quite wild when $X_{\bar{s}}$ has a singularity with embedding dimension greater than 3:

Theorem 4. *Let $X_{\bar{s}}/\bar{s}$ be a proper, integral curve over an algebraically closed field. If $X_{\bar{s}}$ does not lie on a smooth surface (i.e. $X_{\bar{s}}$ has a singularity with embedding dimension at least 3), then $\bar{J}_{X_{\bar{s}}/\bar{s}}$ is reducible.*

Proof. This is proven in both [12] and [11]. The idea can concisely summarized. Let's focus on the case that $X_{\bar{s}}$ is Gorenstein. First, one uses the Abel map to reduce the theorem to show proving the reducibility of the Hilbert scheme $\mathrm{Hilb}^n_{X_{\bar{s}}/\bar{s}}$ for some $n > 0$. Any collection of n distinct points on $X_{\bar{s}}$ defines a point of $\mathrm{Hilb}^n_{X_{\bar{s}}/\bar{s}}$. The Zariski closure of these points is an n-dimensional irreducible component of the Hilbert scheme. Now suppose that p_0 is a singularity of $X_{\bar{s}}$ that has embedding dimension at least 3. One can produce an irreducible component of $\mathrm{Hilb}^n_{X_{\bar{s}}/\bar{s}}$ of dimensional greater than n by studying closed subschemes supported at p_0. In the case of that $X_{\bar{s}}$ is not Gorenstein, the role of the Hilbert scheme must be replaced by an appropriate Quot scheme. □

This is very pathological behavior for a compactification: the boundary is larger than the interior! Thankfully there is a nice characterization of this pathology:

Theorem 5. *Suppose that $X_{\bar{s}}/\bar{s}$ is a proper, integral curve over an algebraically closed field that lies on a smooth surface (i.e. the embedding dimension is at most 2 at every point). Then the compactified Jacobian $\bar{J}_{X_{\bar{s}}/\bar{s}}$ is an integral scheme of dimension g. It is both Cohen-Macaulay and a local complete intersection. Moreover, this scheme contains the Jacobian $J_{X_{\bar{s}}/\bar{s}}$ as a dense open subscheme.*

Proof. In his thesis [5], D'Souza proved this result under the assumption that the singularities of $X_{\bar{s}}$ are either nodes or cusp. Altman, Iarrobino, and Kleiman proved the full theorem as stated in [1]. The irreducibility of $\bar{P}_{X_{\bar{s}}/\bar{s}}$ was also proven by Rego in [12]. Rego proves irreducibility by a careful local study of the compactified Jacobian at a given boundary point.

The proof by Altman, Iarrobino, and Kleiman is quite different. Their proof is global in nature and uses the Abel map to reduce to the case of a Quot scheme. □

There are, of course, many singular curves that lie on surfaces and have singularities that are not double points. Beyond the previous theorem, there seem to be few results describing the global geometry of the compactified Jacobians associated to such curves. One such result is an enumeration of the number of boundary components due to Rego:

Theorem 6. *Suppose that $X_{\bar{s}}/\bar{s}$ is a proper, integral curve over an algebraically closed field that lies on a smooth surface (i.e. the embedding dimension is at most 2 at every point). Let $g : X'_{\bar{s}} \to X_{\bar{s}}$ be the normalization. If $p \in X_s(k_s)$, then let $\delta(p)$ equal the $\dim(g_*(\mathscr{O}_{X'_s})/\mathscr{O}_{X_s}) - 1$. Here $f : X'_s \to X_s$ is the normalization map. The number of irreducible components of the boundary $\partial(\bar{J}_{X_{\bar{s}}/\bar{s}})$ is equal to the sum $\sum \delta(p)$.*

Proof. This is Rego's theorem A [12].

□

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