Extension Criteria

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References: 1) Namikawa's book *Toroidal Compactifications of Siegel Space*, (esp. pages 77-84)

2) Namikawa's thesis, on the canonical map from the moduli space of stable curves to the Igusa monoidal transform

3) Namikawa's paper: A new compactification of the Siegel space and degeneration of abelian varieties.

4) Alexeev & Bruniaret's paper: Extending the Torelli map to toroidal compactifications of Siegel space.

Want to explain a criteria for the Torelli map

t: $\mathcal{M}_g \to \mathcal{A}_g$

to extend to a map

$\tilde{t}: \tilde{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g^*$

from $\tilde{\mathcal{M}}_g$ to a given toroidal compactification of $\mathcal{A}_g$.

To begin, let us observe that the analogous question for the Satake compactification $\mathcal{A}_g^*$ has a very nice answer:

**Thm** The Torelli map extends to a map

$\tilde{t}^*: \tilde{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g^*$.

This is a consequence of a general extension thm for arithmetic quotients of symmetric spaces. Recall

$\mathcal{A}_g$ is the quotient of $D = \mathcal{H}_g$ (a bounded symmetric domain)

by $\Gamma = \text{Sp}(2g, \mathbb{Z})$ (an arithmetic group)
The Satake compactification is defined to be the quotient of $D^* = \text{rational closure}$ by $\Gamma$.

We have the following general extension theorem of Borel-Kobayashi-Ochiai:

**Extension Thm.** Let $D = \text{bounded symmetric domain}$, $\Gamma = \text{arithmetic group acting on } D$.

Then every holomorphic map

$$f : (\Delta^a)^a \times \Delta^b \rightarrow D/\Gamma$$

extends to a regular map

$$\mathfrak{f} : \Delta^a \times \Delta^b \rightarrow (D/\Gamma)^*$$

The proof is an argument by cplx geometry, Siegel sets, arithmetic groups...

The Extension Theorem for the Satake compactification immediately implies that the Torelli map extends. Indeed, the problem of showing that it extends is local on $\overline{M}_g$, and given $[X] \in \overline{M}_g$, there is a neighborhood of $[X]$ of the form $(\Delta^a \times \Delta^b)/\text{finite group}$. Here $a = \Phi(a)$ nodes of $X$ and $b = 3g - 3 - a_1$ and $\Delta^a \times \Delta^b$ is constructed so fix minimal family of curves

$$X_t$$

such that

$$\Delta^a \times \Delta^b \rightarrow \Delta^a \times \Delta^b \times \Delta^3 \times \Delta^5 \times \Delta^\ldots \times \Delta^{3g - 3 - a_1}$$

is the locus where the $i$th node $p_i S_X$ remains a node.

Locally the Torelli map lifts to a map $t : (\Delta^a)^a \times \Delta^b \rightarrow \overline{M}_g$. This lifted map extends by the Extension Theorem. The extended map is invariant under the action of the relevant finite group, so we get the desired extension

$$\overline{M}_g \simeq \frac{\Delta^a \times \Delta^b}{\text{finite group}} \rightarrow \Delta^a / \Gamma$$

**Remark.** The fact that the Torelli map lifts from $\Delta^a \times \Delta^b / \text{finite group}$ to $\Delta^a \times \Delta^b$ is essentially the assertion that the map $t : \overline{M}_g \rightarrow \Delta^a / \Gamma$ lifts to a map out of the coarse moduli space of stable curves.
In this write-up, we will not treat stack-theoretic issues carefully, but it would be nice to have such a write-up as most of the references we are using were written before stacks came into when the theory of stacks was less well-developed.

Remark Namikawa first proved the existence of $t^*$. He did not use the Borel/Kodayashi-Ochiai Extension Theorem, and instead rather proved the result by computing directly with period matrices.

His work shows that if $[X] \in \overline{M}_g$, then $t^*([X]) \in \overline{A}_g \leq A_g^*$ is the product of the Jacobians of the connected components of the normalization $\widetilde{X}$ (i.e., maximal abelian quotient of the generalized Jacobian of $X$).

The Torelli map does not always extend to a map into $A_g^*$, but there is a useful combinatorial criterion for the map to extend.

Recall:

$\text{Ng} = \text{lattice of } g \times g \text{ integral symmetric matrices}$

$= \text{lattice of integral quadratic forms on } \mathbb{Z}^g$

$\text{Mg} = \text{lattice of } g \times g \text{ 1/2-integral symmetric matrices}$

$= \text{lattice of 1/2-integral quadratic forms on } \mathbb{Z}^g$

$x = \text{stable curve}$

$\Gamma_X = \text{dual graph}$

$\text{N}(\Gamma) = \text{lattice of integral quadratic forms on } H_1(\Gamma_X, \mathbb{Z})$

$\text{M}(\Gamma) = \text{lattice of 1/2-integral quadratic forms on } H_1(\Gamma, \mathbb{Z})$

If $e$ is an edge of $\Gamma_X$, we write $e^*$ for the functional $e^*: H_1(\Gamma_X, \mathbb{Z}) \rightarrow \mathbb{Z}$ that sends a chain to the coefficient of $e$.

We then have

\[ \left( e^* \right)^2 \in \text{N}(\Gamma). \]

Example $\Gamma = \begin{array}{c}
\text{v} \\
\text{w}
\end{array}$ Write $\langle \text{(vw)}, (\text{uw})_2, (\text{vw})_3 \rangle$ for the 3 edges oriented from v to w.

$H_1(\Gamma, \mathbb{Z}) = \langle b_1 = (\text{vw})_1 + (\text{uw})_2, b_2 = (\text{vw})_3 \rangle$
The functionals \((\text{vw})^*\) are \(x_1 + x_2\) respectively. The quadratic forms \((\text{vw})^*_1, (\text{vw})^*_2, (\text{vw})^*_3\) are thus \((x_1 + x_2)^2, x_1^2, x_2^2\).

\[ H_1(\Gamma, Z) = \langle b_1 = v_1 w_1 + v_1 w_2 + v_2 w_1 + v_2 w_2 + v_3 w_3, b_2 = v_3 w_1 + v_3 w_2 + v_3 w_3, b_3 = v_3 w_1 + v_3 w_2 + v_3 w_3, b_4 = v_3 w_1 + v_3 w_2 + v_3 w_3 \rangle \]

Quadratic forms are:

\[ (x_1 + x_2)^2, (x_1 - x_2 + x_4)^2, (x_2 - x_3 - x_4)^2. \]

We can now state the Extension Criteria!

Given \(X\), fix a surjection \(s: \mathbb{Z}^g \rightarrow H_1(\Gamma, Z)\). We get \(N(s): N(\Gamma) \rightarrow Ng\) given by \(g \mapsto g s\).

**Extension Criteria**

For a given

\[ \Sigma = \text{admissible fan decomposition} \quad \text{of} \quad \mathcal{C}_{g, s}(Ng)_{\mathbb{R}}. \]

**Thm** The Torelli map \(\overline{M}_g \rightarrow \overline{A}_g^\Sigma\) is regular on a neighborhood of \([X] \in \overline{M}_g \rightarrow \Sigma \in \Sigma\) that contains the images of the forms \((e^s)^*\) under \(N(s)\).

**Example** If \(g = 2\), \(X = \varnothing\) (So \(\Gamma = \varnothing\)), and \(\Sigma\) is the standard fan then it is regular on a neighborhood of \(X\). Indeed, the forms \((e^s)^*\) map to the vertices of \(\partial \sigma\).
The Torelli map \( \tau: \mathcal{M}_g \rightarrow \mathcal{A}_g \) is \textbf{NOT} regular on a neighborhood of \( \Sigma \) if we take \( \Sigma \) to be the barycenter subdivision of the standard fan decomposition.

\[ \frac{1}{2}(x^2 + y^2) - \frac{2}{3}(x^2 + xy + y^2) \]

Here we have a few natural choices for \( \Sigma \).

We have the central - the perfect cone decomposition. This consists of the

\[ \sigma = \text{other principal cone} = \text{cone spanned by } x_1^2, x_2^2, x_3^2, x_4^2, \]
\[ (x_1 - x_2)^2, (x_1 - x_3)^2, (x_1 - x_4)^2, \]
\[ (x_2 - x_3)^2, (x_2 - x_4)^2, (x_3 - x_4)^2, \]
\[ (x_1 - x_2 - x_3)^2, (x_1 - x_2 - x_4)^2, (x_1 - x_2 - x_3 - x_4)^2. \]

and

\[ \tau = \text{cone spanned by } x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_2)^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, (x_3 - x_4)^2, (x_1 + x_2 - x_3)^2, (x_1 + x_2 - x_4)^2, (x_1 + x_2 - x_3 - x_4)^2. \]

\[ = \text{cone dual to ray spanned by perfect form } \frac{1}{2} \left\{ (x_1 - x_2)^2 + x_3^2 + x_4^2 + (x_1 + x_2 + x_3 + x_4)^2 \right\} \]

= cone assos to the \( D_4 \) root system together with faces and translates.

The forms \( (e_i^*)^2 \) do \textbf{not} lie in the principal cone or any of its translates! But it is regular on a neighborhood of \( EX \).

\[ \text{check: the forms } (e_i^*)^2 \text{ are all contained in some translate of } \tau. \]

\[ \text{Alternatively, we could take } \Sigma = 2^{nd} \text{ Voronoi decomposition. This decomposition } \]
The cones are obtained by subdividing $\mathcal{C}$ by adding the central ray $\frac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + \ldots + (x_1 + x_2 + x_3 + x_4)^2)$.

Exercise: Find the two cones and then prove that the $(2^n)^2$-st are contained in one of them.

Remark: Both $\overline{\mathcal{M}_g}$ and $\overline{\mathcal{M}_g}^+$ can be defined as fine stacks or coarse schemes. Presumably the criteria applies to the map $\overline{\mathcal{M}_g} \to \mathcal{M}_g$. All the references I looked only discussed the arithmetic group, but $\Gamma$ is not a neat ($\Rightarrow$ torsion-free).

It would be nice to have the details written down.

Remark: One strange feature of criteria is that one side of the equivalence (but not the other) depends on the choice of a surjection $s: \mathbb{Z}^2 \to H_1(\mathbb{M}_g, \mathbb{Z})$. The reader may easily check that if $s$ is fulfilled for one $s$ then it is fulfilled for all $s$. (Use the GL-equivariance of $\Sigma$.)

The surjection arises naturally in the proof as follows. First, at one point we will pick a rational boundary component representing a given cusp. Second, to analyze a monodromy action, we will pick a splitting of the natural surjection $H_1(X, \mathbb{Z}) \to H_1(\mathbb{M}_g, \mathbb{Z})$.

How to prove the criteria? Given $X$, set $a = \#_a$ of nodes and $b = 3g - 3 - a$.

We can find a family of stable curves $X_t$ s.t.: 1) the fiber over $(a, b)$ is $X$; 2) the locus $\Delta^a \times \Delta^b$ is in the position where the node remains a node; 3) the classifying map defines an open immersion $\Delta^a \times \Delta^b \to \mathcal{M}_g$. 

\[
\Delta^a \times \Delta^b \to \overline{\mathcal{M}_g}.
\]
So we are in the same situation as when we were studying the map into \( \Lambda^g \). The general problem of extending a map \((\Lambda^*)^q \times \Delta^b \to D/\Pi\) to a map \(\Delta^a \times \Delta^b \to (D/\Pi)^\Sigma\) is analyzed in Chapter III, Section 7. (except they only treat the case where \( \Pi \) is neat). First, there is a lift

\[
\begin{align*}
Z^q &= \Pi_i (\Lambda^*)^q \times \Delta^b) \\
\exp &\downarrow \quad \sim \downarrow \\
H^a \times \Delta^b &\to D / \Pi \\
\end{align*}
\]

This existence of the lift follows from covering space theory when \( \Pi \) is neat; and probably follows more; the lift probably exists when \( \Pi \) is neat, as well as a map into the quotient stack \([D/\Pi]\).

**Proposition 7.1 of AMRT states:**

There is an exact sequence

\[
\begin{align*}
(H_i^*)^q \times \Delta^b &\to (F)^* \\
\text{rat}^0 \text{ closure} \\
\end{align*}
\]

Set \( p = (F)^* (1, 0, \ldots, 1, 0) \). The point lies in a unique rad-nil boundary component \( F \). Asso to \( F \) are

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

The cone \( \overline{C(F)} \) gets a fan decomposition \( \Sigma(F) \) by pulling back to \( \Sigma \) via an embedding \( C(F) \subset \Sigma \).
Set \[ \Lambda'(F) = \mathcal{P}(F) \cap \Gamma', \quad \Lambda''(F) = U(F) \cap \Gamma', \quad \gamma_1, \ldots, \gamma_n = \text{image of } \operatorname{Stm}. \]

Check: \( \gamma_1, \ldots, \gamma_n \in \Lambda'(F) \), so we have an induced map \[ f^0: (\Delta^a) \times \Delta^b \rightarrow D \backslash \Lambda'(F) = X(F). \]

Recall \( X(F) \hookrightarrow \mathcal{X}(F) = T(F) \times V(F) \times H(F) \)

\[ \begin{array}{c}
\cap \\
\cap \\
\cap \\
\text{cplx torus} \\
\text{cplx vector space} \\
\Sigma \\
\text{Siegel half-plane}
\end{array} \]

\( X_\Sigma(F) \hookrightarrow \mathcal{X}_\Sigma(F) = T_\Sigma(F) \times V(F) \times H(F) \)

The space \( \mathcal{X}_\Sigma(F) \) has a natural action of \( \Lambda'(F) \) and \( \tilde{\Lambda}'(F) \).

The following result shouldn't be too surprising:

**Thm. (1/2 of Thm 7.2 in AMRT)**

\[ f: (\Delta^a) \times \Delta^b \rightarrow D \gamma_1 \text{ extends to } \]

\[ f: \Delta^a \times \Delta^b \rightarrow (\Delta^a)^\Sigma \]

\[ f^0: (\Delta^a)^{\times n} \times \Delta^b \rightarrow X(F) = D \Lambda'(F) \text{ extends to } \]

\[ f^0: \Delta^a \times \Delta^b \rightarrow X_\Sigma(F). \]

The map \( f^0 \) is essentially a map of trivial torus bundles, and we're essentially a map of trivial toric bundles, so we'd expect to be able to analyze this problem combinatorially.

In fact, we have \( \chi \in \mathcal{C}(F) \subseteq U(F) \) and

**Thm** \( f^0 \) extends to \( f^0: \sigma \in \Sigma(F) \) s.t. all the \( \chi \)'s he in \( \sigma \).

We can choose \( F = \text{std. boundary component} \) in which case \( U(F) \) is identified with \( (N_\gamma')_\mathbb{R} \). \( \Sigma(F) \) is the pullback of \( \Sigma \) under the embedding \( U(F) \hookrightarrow (N_\gamma')_\mathbb{R} \).

We choose to establish the criteria we need to compute the elements \( \gamma_{n-1}, \gamma_n \).
The map \( \Phi: \mathbb{Z}^q \to \pi_1((\Delta^k \times \Delta^l)^0) \to \text{Sp}(2g, \mathbb{Z}) \) has the following natural description: given \( X \in \pi_1((\Delta^k \times \Delta^l)^0) \), monodromy induces an pairing. If we fix a basis for \( H_1(X_0, \mathbb{Z}) \) then we can represent this automorphism by an element of \( \text{Sp}(2g, \mathbb{Z}) \), and this element is \( \phi(X) \). (Note: Since \( \pi_1 \) is abelian, \( \phi(X) \) is independent of our choice of basis.)

We can compute \( \phi(X) \) using the Picard–Lefschetz formula:

**Picard–Lefschetz Formula**

Let \( X_t \to \Delta \) be a family of curves with
- total space \( X_t \) smooth;
- \( X_0 \) a curve with 1 node;
- \( X_t \) smooth for \( t \neq 0 \).

Fix a base point \( \ast \in \Delta^k \). Then \( \exists \)

\[ c_{\text{van}} \in H_1(X_0, \mathbb{Z}) \]  
("vanishing cycle") s.t. the monodromy operator \( H_1(X_t, \mathbb{Z}) \to H_1(X_{t_0}, \mathbb{Z}) \) given by the action of the std. generator is given by:

\[ C \to C + \langle c, c_{\text{van}} \rangle \cdot c_{\text{van}} \]  
("transvection across line \( \mathbb{R} \cdot c_{\text{van}} \)"

**Picture**

\[ X_t \]  
\[ X_0 \]  
\[ \text{after applying monodromy} \]
The ele. $Y_i$ has the following description. Pick a general arc $\Delta \to \Delta^a \times \Delta^b$ that passes through $v$ and meets $\Delta^x \times \Delta^0 \times \Delta^y \times \Delta^z$ transversely at $0$.

Consider the restriction of the

$$\nu_{\Delta^1}$$

miniversal family. The total space might be singular, but we can resolve singularities $Y_t \to X_t / \Delta$ w/o changing the monodromy. Then $Y_i = \text{transvers} \text{ion across the line spanned by the vanishing cycle associated to}$

$Y_t \to \Delta$.

We can compute these vanishing cycles using a particularly nice basis for $H_1(X_t, \mathbb{Z})$. Pick a family of cycles on $X_t \to (\Delta^1)^a \times \Delta^b$ s.t:

a) $\{ A_1(t), \ldots, A_g(t), B_1(t), \ldots, B_g(t) \}$ is a standard basis on every fiber.

b) $\lim_{t \to 0} A_1(t), \ldots, A_g(t), B_1(t), \ldots, B_g(t) = \text{standard basis for } H_1(\tilde{X}_0, \mathbb{Z})$

c) $\lim_{t \to 0} B_g(t), \ldots, B_g(t) = \text{normalization}$

d) $\lim_{t \to 0} A_g(t) = 0$

By (d), every vanishing cycle can be written as

$$C_{\text{van}} = e_1 A_{g_1} + \ldots + e_h A_g \quad \text{w/ } h = g - g'$$
So if the vanishing cycle assoc. to the $i$th node $p_i$ is $C_{van} = \sum c_i A_{i+g}$, then $\gamma_i$ acts by fixing $A_{n-g}$ and $B_{n-g}$, and sending $B_{g+1} \rightarrow B_{g+1} + (B_{g+1}, C_{van}) C_{van}$

\[
\begin{align*}
&\rightarrow \quad B_{g+1} + c_1 A_{g+1} + c_2 A_{g+2} + \ldots + c_{n-g} A_{n-g} \\
&\rightarrow \quad B_{g} + c_1 A_{g} + c_2 A_{g+1} + \ldots + c_{n-g} A_{n-g}.
\end{align*}
\]

In other words, $\gamma_i$ is given by the matrix

\[
\begin{pmatrix}
\text{Id} & c_1 c_1 & c_2 c_1 & \ldots & c_{n-g} c_1 \\
& c_1 c_2 & c_2 c_2 & \ldots & c_{n-g} c_2 \\
& & \ddots & \ddots & \vdots \\
& & & c_1 c_{n-g} & c_2 c_{n-g} & \ldots & c_{n-g} c_{n-g}
\end{pmatrix}
\]

This is the matrix of the quadratic form $(c_1 x_1 + \ldots + c_{n-g} x_{n-g})^2$. So to verify the criteria, one needs to check: the functional $e_i^*: H_1(\Gamma_X, \mathbb{Z}) \rightarrow \mathbb{Z}$ is the functional satisfies

\[e_i^*(B_j(a)) = (B_j, C_{van})\]

maybe there should be a sign here.

**Last remark:** In general, the quadratic forms $(e_i^*)^2$ can be difficult to describe. However, when $\Gamma_X$ is planar, the form $\omega$ has a simple description: every functional $e^*$ can be written as $e^* = 0$ or $e^* = \alpha x_i$ for some $i$, or $e^* = x_i - x_j$ for some $i \neq j$. So these forms lie in the principal cone!