This work is joint with:

- Sebastian Casalaina-Martin (University of Colorado at Boulder);
- Filippo Viviani (Roma Tre).

Alternative approaches to results are given by:

- the Chai-Faltings-Mumford theory of Uniformization (Alexeev and Nakamura);
- the theory of the Presentation Scheme (Oda and Seshadri).

We will work over the complex numbers $k := \mathbb{C}$.  
(But ask if you are curious about a more general $k$!)
The Jacobian variety

To a non-singular curve $X$ of genus $g$, one can associated the Jacobian:

$$J^d_X = \text{the Jacobian variety,}$$

$$= \text{the moduli space of (degree } d) \text{ line bundles}$$

$$= \text{a complex torus}.$$
The Jacobian is a basic tool for studying $X$.  

**Theorem (Torelli)**

If

\[ J^0(X) \cong J^0(Y) \ (\text{with polarization}), \]

then

\[ X \cong Y. \]
Question (Mayer and Mumford, 1964)

1. Is there an analogue when $X$ is nodal?
2. If yes, do they fit into a family over $\overline{M}_g$?

Will focus on Question 1 for a specific curve.

Only generalize $J^d_X$ for $d = 0$.

Write $\overline{J}^0_X$ for analogue of $J^0_X$. 
Draw Picture of Genus 3 curve whose dual graph is 2 vertices joined by 4 edges.
Attempt One

Form the moduli space of degree 0 line bundles!

Fails! This does not give a well-behaved scheme.

In genus 3 example, have new invariant:

$$\text{bidegree of } L = (\deg L|_{X_1}, \deg L|_{X_2})$$
Draw infinite collection of copies of $\bigoplus_{i=1}^{3} \mathbb{C}^{\ast}$ indexed by possible bidegrees.
Attempt One: Problems

The problems are:

- the moduli space is NOT of finite type;
- the moduli space is NOT universally closed;
- more problems in a family (NOT separated).
Construct as a GIT quotient of a Quot scheme!

Assume $d \gg 0$. Form

$$U = \{(L; s_1, \ldots, s_r) : s_1, \ldots, s_r \in H^0(L) \text{ basis}\},$$

and the natural compactification

$$\text{Quot}(X, O^r) \supset U.$$
Attempt Two: Construction

Have (linearized) action of $\text{SL}_r$ given by change of basis.

Form GIT quotient

$$\overline{J}_X^d := \text{Quot}(X, \mathcal{O}^r) \sslash \text{SL}_r.$$
Hard part: How to interpret points of $\overline{J}_X^d$?

**Theorem (Caporaso-Pandharipande-Simpson)**

*The scheme $\overline{J}_X^d$ is a coarse moduli space of slope-stable rank 1, torsion-free sheaves.*

Lots of generalizations. I know 10(!) other papers on this subject.
In genus 3 example, $\mathcal{J}^0(X)$ has 3 irreducible components.

Parameterizes line bundles of bidegree

$$(-2, 2), (-1, 1), (0, 0), (1, -1), (2, -2).$$

and their degenerations.

**Question**

What is the local structure of $\mathcal{J}^0_X$ at $l := f_*(\mathcal{O}_X(-2, -2))$? How many local components?
Proposition (Example)

There is isomorphism

\[ \text{the completed local ring of } \tilde{J}_X^0 \text{ at } I \cong R^H, \]

where

\[ H := \text{Aut}(I)/\{\text{scalars}\} \]
\[ = (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{C}^* \]

acting on

\[ R := \bigotimes_{i=1}^4 \mathbb{C}[[u_i, v_i]]/(u_i v_i) \]
\[ = \text{the miniversal deformation ring for } I. \]
Proposition

The group action is

\[ u_i \xrightarrow{(a,b)} ab^{-1} u_i, \]
\[ v_i \xrightarrow{(a,b)} ba^{-1} v_i. \]

Proof.

There are three inputs:

- Luna’s Slice Theorem shows an action exists;
- Rim’s Theorem shows action is unique;
- compute using deformation theory.
Theorem (C.-M., K., V.)

Let $I$ be a polystable rank 1, torsion-free sheaf on a nodal curve $X$ that fails to be locally free at a set $\Sigma \subset X$. Then the completed local ring of $\overline{J}^d(X)$ at $I$ is isomorphic to a power series ring over the completed cographic ring of $\Gamma_X(\Sigma)$. 
In the example, the theorem states that the ring is generated by

\[ x_1 y_2, \quad x_1 y_3, \]
\[ x_1 y_4, \quad x_2 y_3, \]
\[ x_2 y_4, \quad x_3 y_4, \]
\[ y_1 x_2, \quad y_1 x_3, \]
\[ y_1 x_4, \quad y_2 x_3, \]
\[ y_2 x_4, \quad y_3 x_4, \]

which correspond to oriented cycles.
How many local components in the genus 3 example? The answer is:

$$14 = 8 + 6.$$ 

Which correspond to totally cyclic orientations.
Thank you!