CHOW RING OF A BLOW-UP

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1. Introduction

In this document, we compute the Chow ring of the blow-up of $\mathbb{P}^3$ along the twisted cubic. Our goal is to illustrate the main results of §6.7 of [Ful84] with a concrete example. We begin by fixing notation.

(1) X denotes projective space $\mathbb{P}^3$,
(2) Z denotes the twisted cubic curve in X,
(3) $\tilde{X}$ denotes the blow-up of X along Z,
(4) $\tilde{Z}$ denotes the exceptional divisor of the blow-up.

If we forget about the embedding in X, then Z is just the projective line.

These three schemes fit into the following Cartesian diagram:

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{j} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array} \tag{1.1}$$

Our goal is to give a complete description of the Chow ring $\mathrm{Ch}^*(\tilde{X})$.

Following Fulton, we describe the additive structure of $\mathrm{Ch}^*(\tilde{X})$ in terms of the groups $\mathrm{Ch}^*(X)$, $\mathrm{Ch}^*(\tilde{Z})$, and $\mathrm{Ch}^*(Z)$, together with the Chern classes of the normal bundle $N := N_{Z/X}$. Having described the additive structure, we then compute the multiplicative structure using some standard intersection theory tools.

The Chow rings of X and Z are well-known. They are given by

$$\mathrm{Ch}(X) = \mathbb{Z}[h]/(h^4), \tag{1.2}$$

$$\mathrm{Ch}(Z) = \mathbb{Z}[w]/(w^2). \tag{1.3}$$

Here h is the class of a hyperplane and w is the class of a point. The maps i_* and i^* are also easy to describe. The subvariety Z is the image of a map $\mathbb{P}^1 \to X$ given by cubic polynomials, and it is easy to deduce the following equations from that fact:

$$i_*(1) = 3h^2, \tag{1.4}$$

$$i_*(w) = h^3, \tag{1.5}$$

$$i^*(h) = 3w. \tag{1.6}$$

We begin by describing the ring $\mathrm{Ch}^*(\tilde{Z})$.

2. Geometry of the Exceptional Divisor

The blow-down map $g\colon \tilde{Z} \to Z$ realizes the exceptional divisor as a projective bundle over Z. More precisely, the scheme $\tilde{Z}$ is the projectivized normal bundle $\mathbb{P}(N)$. The geometry of $\tilde{Z}$ can thus be completely described in terms of the Chern classes of N.

As both X and Z are non-singular, N is the quotient of $T(X)|_Z$ by $T(Z)$. The Chern classes of these tangent bundles can be computed from the Euler exact sequence. This computation shows that

$$c(N) = 1 + 10w, \tag{2.1}$$

and we can conclude that the Chow ring of $\tilde{Z}$ may be described by

$$\mathbb{Z}[w, z]/(w^2, z^2 + 10wz). \tag{2.2}$$

Here we are abusing notation and writing $w \in A^1(\tilde{Z})$ for the pull-back $g^*(w)$, and we will frequently use this notation for classes. Concretely, $w \in A^1(\tilde{Z})$ is the class of a fiber of $\tilde{Z} \to Z$. The class z is the Chern class $c_1(\mathcal{O}(1))$ of the relative Serre bundle.

3. The Additive Structure of $\mathrm{Ch}^*(\tilde{X})$

Having described the Chow ring of $\tilde{Z}$, we can almost immediately describe the additive structure of $\mathrm{Ch}^*(\tilde{X})$. The additive structure is described by Proposition 6.7 of [Ful84]. Recall that there is a short exact sequence

$$\mathrm{Ch}_k(Z) \hookrightarrow \mathrm{Ch}_k(\tilde{Z}) \oplus \mathrm{Ch}_k(X) \twoheadrightarrow \mathrm{Ch}_k(\tilde{X}) \tag{3.1}$$

for every integer k. The injective map is given by $x \mapsto (c_1(M) \cap g^*(x), -i_*(x))$, while the surjective map is given by $(\tilde{x}, y) \mapsto j_*(\tilde{x}) + f^*(y)$. Here E denotes the excess normal bundle.

Recall that the excess normal bundle is defined to be the quotient bundle $M := g^*(N)/\mathscr{O}(-1)$. The Chern classes are given by

$$\begin{aligned} c(E) &= (1 + 10w)/(1 - z) \\ &= (1 + 10w)(1 + z + z^2) \\ &= 1 + 10w + z + z^2 + 10wz \\ &= 1 + 10w + z. \end{aligned}$$

We now have enough information to describe the additive structure of the Chow ring. Let $e \in \mathrm{Ch}_2(\tilde{X})$ denote the class of the exceptional divisor and $h \in \mathrm{Ch}_2(\tilde{X})$ denote the image of the hyperplane class under f^*. As additive groups, the Chow groups can be described as

1. $\mathrm{Ch}_3(\tilde{X})$ is freely generated by the fundamental class 1;
2. $\mathrm{Ch}_2(\tilde{X})$ is freely generated by h and e;
3. $\mathrm{Ch}_1(\tilde{X})$ is generated by the elements w, z, and h^2, which satisfy the relation $10w + z = 3h^2$;
4. $\mathrm{Ch}_0(\tilde{X})$ is freely generated by h^3.

Both w and z are the images of the appropriate classes under j_*, so the geometric meaning of these classes should be clear. On the other hand, the class h is the image of the hyperplane class under f^*, and in general, the geometric meaning of pulling back via a blow-up map is slightly obscure. Say that $V \subset X$ denotes a general hyperplane. If $\tilde{V}$ is the proper transform, then the class h can be computed as

$$h = [\tilde{V}] + j_* \left\{ c(M) \cap g^* s(V \cap Z, V) \right\}_2 . \tag{3.2}$$

The scheme-theoretic intersection $V \cap X$ consists of 3 points, so $g^*s(V \cap Z, V)$ is supported on the union of 3 fibers of $\tilde{Z} \to Z$, and hence the second term in Equation (3.2) must be zero for dimensional reasons. We can conclude that the class h is just the class of the proper transform of a hyperplane.

Another obvious subvariety of $\tilde{X}$ that can be associated to V is the total transform. This variety is the union of the proper transform $\tilde{V}$ and 3 rational curves. In particular, the variety is not pure dimensional, and hence does not have a natural class in $\mathrm{Ch}_2(\tilde{X})$. We now turn out attention to the multiplicative structure.

4. The Multiplicative Structure of $\mathrm{Ch}^*(\tilde{X})$

We describe the multiplicative structure of $\mathrm{Ch}^*(\tilde{X})$ by describing the intersection pairing between curves and divisors. The pairing can be described by the following matrix:

$$\begin{array}{c|ccc} & w & z & h^2 \\ \hline e & -1 & 10 & 0 \\ h & 0 & 3 & 1 \end{array} \tag{4.1}$$

How are the intersection numbers computed? The numbers

$$\int_{\tilde{X}} wh, \int_{\tilde{X}} h^2 e, \int_{\tilde{X}} h^2 h$$

can be computed by elementary considerations. Observe that, for example, a general line in X is disjoint from the twisted cubic Z.

The computation of the number zh is an application of the adjunction formula. We have

$$\begin{aligned} \int_{\tilde{X}} zh &= \int_{\tilde{X}} j_*(c_1(\mathcal{O}(1))) \cap f^*h \\ &= \int_{\tilde{Z}} c_1(\mathcal{O}(1)) \cap j^* f^* h \\ &= \int_{\tilde{Z}} c_1(\mathcal{O}(1)) \cap g^* i^* h \\ &= \int_{\tilde{Z}} c_1(\mathcal{O}(1)) \cap 3w \\ &= 3. \end{aligned} \tag{4.2}$$

The remaining intersection numbers can be computed using the excess intersection formula. For example we have

$$\begin{aligned} \int_{\tilde{X}} ew &= \int_{\tilde{X}} j_*(1) j_*(w_0) \\ &= \int_{\tilde{Z}} j^* j_*(w_0) \\ &= \int_{\tilde{Z}} c_1(\mathcal{O}(-1)) \cap w_0 \\ &= -1. \end{aligned} \tag{4.3}$$

The computation of ze is similar.

5. Applications

Let us now compute some Chow classes in terms of the bases that we have exhibited. There is a net of space quadrics that contain the twisted cubic Z. Fix a general such quadric Q, and let $\tilde{Q}$ equal the proper transform of Q and $\tilde{q}$ the class of $\tilde{Q}$. As a subvariety of Q, the twisted cubic is divisor, and so the blow-down map $\tilde{Q} \to Q$ is an isomorphism. What is the class q in terms of the basis e, h?

We can compute the class using the Blow-Up Formula (Theorem 6.7) from [Ful84]. Using the fact that $f^*([Q]) = 2h$, we have

$$2h = q + j_* \left\{ c(M) \cap g^* s(Q \cap Z, Q) \right\}_2 . \tag{5.1}$$

The scheme-theoretic intersection $Q \cap Z$ is just the twisted cubic Z. Considered as a divisor on Q, the twisted cubic is the divisor of type $(2, 1)$. The normal bundle to Z in Q is trivial, and so $s(Q \cap Z, Q) = c(N_{Z/Q})^{-1} = 1$. We can compute the second term in (5.1). It is:

$$c(M) \cap g^* s(Q \cap Z, Q) = c(M) \cap 1 \\ = 1 + 10w + z. \tag{5.2}$$

We can conclude that $q = 2h - e$. As an aside, this computation shows that $f^*[Q]$ is the class of the total transform of the quadric.

We could also try to compute the class q by computing two of the three intersection numbers

$$\int_{\tilde{X}} wq, \int_{\tilde{X}} zq, \int_{\tilde{X}} h^2 q.$$

The first and last of these numbers is the easiest to compute. Indeed, we have

$$\int_{\tilde{X}} h^2 q = 2$$

since a general line meets Q at 2 points that do not lie on Z. The first number can be given by elementary considerations as well. Considered as a subvariety of $\tilde{Q}$, the scheme-theoretic intersection $\tilde{Q} \cap \tilde{Z}$ is just the twisted cubic. If we instead consider $\tilde{Q} \cap \tilde{Z}$ as a subvariety of $\tilde{Z}$, then this locus is a section of $g \colon \tilde{Z} \to Z$, and so $\tilde{Q}$ meets a fiber of g in a single point. Glossing over issues of transversality, we have that

$$\int_{\tilde{X}} wq = 1.$$

The class of $\tilde{Q} \cap \tilde{Z}$ in $\mathrm{Ch}_1(\tilde{Z})$ is $j^*(q) = 6w + e$ and we see that $6w + e$ is the class of a section of $\tilde{Z} \to Z$.

From the determination of the class q, we can compute the self-intersection number as

$$\begin{aligned} \int_{\tilde{X}} q^3 &= \int_{\tilde{X}} (2h - e)^3 \\ &= \int_{\tilde{X}} 8h^3 + -12h^2 e + 6he^2 + -e^3 \\ &= 8 + 0 + 18 + -10 \\ &= 16. \end{aligned} \tag{5.3}$$

References

[Ful84] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984.

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