A CLASSICAL PROOF THAT THE ALGEBRAIC HOMOTOPY CLASS OF A RATIONAL FUNCTION IS THE RESIDUE PAIRING

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ABSTRACT. C. Cazanave has identified the naive algebraic homotopy class of a rational function of 1 variable with an explicit nondegenerate symmetric bilinear form. Here we relate Cazanave’s result to classical results and in particular identify Cazanave’s form with a residue pairing from commutative algebra. We announce work identifying the local A1-degree of a polynomial map in several variables that has an isolated zero at the origin with the Eisenbud–Levine–Khimshiashvili form.

In this paper we relate Cazanave’s description of the algebraic homotopy class of a rational function to classical results and, in particular, relates his description to the residue pairing. Cazanave’s description is closely related to the following result of Hurwitz. A real rational function \( f(x)/g(x) \) defines a continuous map \( F : P^1_\mathbb{R}(\mathbb{R}) \to P^1_\mathbb{R}(\mathbb{R}) \) from the 1-sphere \( S^1 \cong P^1_\mathbb{R}(\mathbb{R}) \) to itself, and Hurwitz’s result computes the topological degree as

\[
\text{deg}(F) = \text{signature of Béz}(F).
\]

Here \( \text{Béz}(F) \) is the Bézout matrix, an explicit symmetric matrix (see (3) in Section 1).

A reformulation of Hurwitz’s result is that \( \text{Béz}(F) \) determines the homotopy class \([F] \in [P^1_\mathbb{R}(\mathbb{R}), P^1_\mathbb{R}(\mathbb{R})]\) because the degree is the only homotopy invariant. Cazanave’s result is an enrichment of this result to a determination of an algebraic (or motivic) homotopy class. Over an arbitrary field \( k \), two important substitutes for \([P^1_\mathbb{R}(\mathbb{R}), P^1_\mathbb{R}(\mathbb{R})]\) are the set \([P^1_k, P^1_k]^N\) of naive homotopy classes and the set \([P^1_k, P^1_k]^\text{A1}\) of A1-homotopy classes.

In celebrated work describing \([P^1_k]^\wedge n, (P^1_k)^\wedge n)^\text{A1}\) for any \( n \), Morel showed that \([P^1_k, P^1_k]^\text{A1}\) is closely related to symmetric bilinear forms by constructing an isomorphism between \([P^1_k, P^1_k]^\text{A1}\) and a certain extension of the Grothendieck–Witt group that we recall below. Cazanave proved that this isomorphism can be described as the isomorphism that sends \( F \) to the class of its Bézout matrix and proved a parallel result for \([P^1_k, P^1_k]^N\).

Here we interpret Cazanave’s description in the theory of duality for 0-dimensional rings. If \( F \) is defined by a fraction \( f(x)/g(x) \) in lowest terms with \( f \) monic and satisfying \( \text{deg}(f) > \text{deg}(g) \), then the algebra \( Q(F) := k[x, 1/g]/(f/g) \) carries a bilinear pairing, the residue pairing, and we show Cazanave’s result can be interpreted as

**Main Theorem.** The algebraic homotopy class of \( F : P^1_k \to P^1_k \) is represented by the Gram matrix of the residue pairing with respect to the basis dual to the basis \( 1/g(x), x/g(x), x^2/g(x), \ldots \).

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In the classical work related to Hurwitz’s result, the Bézout matrix is studied along with several other matrices. We recall the definition the Bézout matrix and the other matrices in Section 1. We identify those matrices with Gram matrices in Lemma 3 of Section 2, and then in Corollary 7 of Section 3, we explain which matrices represent the homotopy class of $F$.

To prove the Main Theorem, it is enough, by Cazanave’s work, to identify $\text{Béz}(F)$ as the appropriate Gram matrix, and we establish this identification, but we also give an independent proof of a weakening of the Main Theorem, namely that the stable homotopy class of $F$ is the class of $\text{Béz}(F)$. Recall that two rational functions are stably homotopic if they have the same image in $\varprojlim (\mathbb{P}^1_k)^{\wedge_n, (\mathbb{P}^1_k)^{\wedge n}}\mathbb{A}^1$, where the limit is taken with respect to the transition maps $((\mathbb{P}^1_k)^{\wedge n}, (\mathbb{P}^1_k)^{\wedge n})\mathbb{A}^1 \to ((\mathbb{P}^1_k)^{\wedge n+1}, (\mathbb{P}^1_k)^{\wedge n+1})\mathbb{A}^1$ given by smash product with $\text{id} : \mathbb{P}^1_k \to \mathbb{P}^1_k$. For rational functions, being stably homotopic is a strictly weaker condition than being homotopic (in contrast to the situation for continuous maps from the 1-sphere $S^1$ to itself). Our proof of the result is modeled on textbook proofs of Hurwitz’s result. The fact that our technique computes the stable homotopy class but not the unstable homotopy class illuminates the relation of Hurwitz’s result to $\mathbb{A}^1$-homotopy theory. We now explain this point, beginning with a more precise description of the work of Cazanave and Morel.

The set $\{\mathbb{P}^1_k, \mathbb{P}^1_k\}^N$ described by Cazanave is the set of pointed naive homotopy classes of pointed rational functions. This is the quotient of the set of pointed rational functions by the equivalence relation obtained by defining a pointed naive homotopy to be a morphism $H : \mathbb{P}^1_k \times_k \mathbb{A}^1_k \to \mathbb{P}^1_k$ satisfying $H(\infty, t) = \infty$. Cazanave proved that the assignment $F \mapsto \text{Béz}(F)$ defines a bijection between $\{\mathbb{P}^1_k, \mathbb{P}^1_k\}^N$ and the set of pairs $(w, d)$ consisting of a stable isomorphism class $w$ of a nondegenerate symmetric bilinear form and a scalar $d \in k^*$ representing the discriminant $\text{disc}(w) \in k^*/(k^*)^2$ [Caz12, Proposition 3.9].

Furthermore, Cazanave proved that this result is compatible with Morel’s description of $\{\mathbb{P}^1_k, \mathbb{P}^1_k\}^{A^1}$, the group of unstable pointed $A^1$-homotopy classes of pointed rational functions. In contrast to $\{\mathbb{P}^1_k, \mathbb{P}^1_k\}^{N}$, the $A^1$-homotopy group $\{\mathbb{P}^1_k, \mathbb{P}^1_k\}^{A^1}$ is not defined in an elementary manner. It is defined as a by-product of Morel and Voevodsky’s construction of the $A^1$-homotopy category of smooth $k$-schemes [MV99], and has a natural group structure. Morel described this group by constructing a degree homomorphism

$$\text{deg}^{A^1} : \{\mathbb{P}^1_k, \mathbb{P}^1_k\}^{A^1} \to GW(k)$$

to the Grothendieck–Witt group $GW(k)$ of nondegenerate symmetric bilinear forms that induces an isomorphism with the set of pairs $(w, d)$ consisting of a class $w \in GW(k)$ and a scalar $d \in k^*$ that represents the discriminant ([Mor12, Theorem 7.36] for $k$ perfect; see [Mor12, Remark 1.17] and [Hoy14, Footnote 1] for $k$ imperfect). Cazanave proved that this description of $\{\mathbb{P}^1_k, \mathbb{P}^1_k\}^{A^1}$ is compatible with his description of $\{\mathbb{P}^1_k, \mathbb{P}^1_k\}^{N}$ [Caz12, Proposition 3.9], and in particular, showed

$$\text{(2)} \quad \text{deg}^{A^1}(F) = [\text{Béz}(F)] \text{ in } GW(k).$$

The real realization of (2) (in the sense of [MV99, Section 3.3]) is Hurwitz’s result (1). The proof of (1) thus obtained from Cazanave’s work is different from the textbook proofs.
of Hurwitz’s result. In Cazanave’s work, a key role is played by a certain monoid structure on $[\mathbb{P}_k^1, \mathbb{P}_k^1]^N$ that Cazanave constructed. For example, he proves the result about the naive homotopy class by showing that if $F \mapsto \text{Béz}(F)$ defines a monoid homomorphism $[\mathbb{P}_k^1, \mathbb{P}_k^1]^N \to GW(k)$ that preserves a natural grading and then making use of a known description of $GW(k)$ in terms of generators and relations.

In the textbook proofs of Hurwitz’s result (the relevant literature is discussed at the end of Section 1), a monoid structure does not play a prominent role. Instead, a proof modeled on the textbook proofs runs as follows. The topological degree of $F$ equals the sum, over the real zeros of $F$, of the local topological degrees. To identify this sum with the signature of $\text{Béz}(F)$, we pass to the complex numbers. Over the complex numbers, $\text{Béz}(F)$ is $GL$-equivalent to a block diagonal matrix $\text{New}(F)$ with blocks indexed by the complex roots of $f(x)$. We complete the proof by computing the signature of $\text{Béz}(F)$ in terms of the blocks, computing that a pair of blocks corresponding to a complex conjugate pair of roots contributes 0 to the signature and a block corresponding to a real root contributes to the signature the local degree at the root.

We show that the same argument, suitably modified, proves that the stable homotopy class of $F$ is represented by $\text{Béz}(F)$, but unless Cazanave’s result is used, the argument does not prove that $\text{Béz}(F)$ represents the unstable homotopy class. The issue is that, while $\deg^{\mathbb{A}^1}(F)$ can be computed as a sum over $F^{-1}(0)$, the unstable homotopy class cannot be computed from $F^{-1}(0)$ in an analogous manner. We demonstrate this by explicit example at the end of Section 3, and then we explain how the argument of this paper could be extended if a formula, Equation (13) and its generalization, could be proven independently of Cazanave’s work.

Finally, the Main Theorem is related to the beautiful signature theorem of Eisenbud–Levine and Khimshiashvili, which identifies the local degree (or index) of a real polynomial function as the signature of the residue pairing. In Section 4 we discuss that work and its relation to the Main Theorem. In particular, we announce the following theorem which generalizes the signature theorem and answers a question posed by Eisenbud in [Eis78]:

**Theorem 1** (The main result of [KW]). The local $\mathbb{A}^1$-degree of $f$: $\mathbb{A}^n_k \to \mathbb{A}^n_k$ is the Eisenbud–Khimshiashvili–Levine class $w(\beta_0)$.

**Conventions**

A pointed rational function is a nonconstant $k$-morphism $F: \mathbb{P}_k^1 \to \mathbb{P}_k^1$ satisfying $F(\infty) = \infty$. Every such morphism can be represented by a fraction $f(x)/g(x) \in \text{Frac} k[x]$ with $f$ monic, $f$ relatively prime to $g$, and $\deg(f) > \deg(g)$.

Given two pairs $(V_1, \beta_1), (V_2, \beta_2)$ each consisting of a $k$-vector space $V_i$ and a nondegenerate bilinear form $\beta_i$, we say that $(V_1, \beta_1)$ is stably isomorphic to $(V_2, \beta_2)$ if there exists a third such pair $(V_3, \beta_3)$ such that $(V_1 \oplus V_3, \beta_1 \oplus \beta_3)$ is isomorphic to $(V_2 \oplus V_3, \beta_2 \oplus \beta_3)$. Stably isomorphic pairs are isomorphic provided the characteristic of the ground field is not 2.
The Gram matrix of a symmetric bilinear form $\beta$ on a finite dimensional $k$-vector space $V$ with respect to a basis $e_1, e_2, \ldots, e_\mu$ is the matrix $(\beta(e_i, e_j))$.

We write $S(k)$ for the set of nondegenerate symmetric matrices with entries in $k$.

1. THE MATRICES

Here we recall the definition of the symmetric matrices associated to a rational function that Kreĭn–Naĭmark discuss in their survey of matrix methods for analyzing the zeros of a real polynomial [KN81]. There are four such matrices, including the Bézout matrix. In Section 2 we relate these matrices to the residue pairing and then in Section 3 we show that all four matrices represent the stable homotopy class of $F$ and some but not all represent the unstable homotopy class.

The Bézout matrix $\text{Béz}(F)$ of a pointed rational function is defined as follows: If $F : P^1_k \to P^1_k$ is defined by a fraction $f(x)/g(x) \in \text{Frac} k[x]$ written in lowest terms with $f$ is monic and satisfying $\deg(f) > \deg(g)$, then write

$$f(x)g(y) - f(y)g(x) = (x - y) \cdot \left( \sum_{i,j=1}^{\mu} b_{i,j} x^{i-1} y^{j-1} \right)$$

in $k[x, y]$, with $\mu$ the degree of $f$ and define

$$\text{Béz}(F) := \begin{pmatrix} b_{1,1} & \cdots & b_{1,\mu} \\ \vdots & \ddots & \vdots \\ b_{\mu,1} & \cdots & b_{\mu,\mu} \end{pmatrix} \in S(k).$$

The matrix $S(F)$ is defined by writing

$$g(x)/f(x) = s_1/x + s_2/x^2 + s_3/x^3 + \ldots \text{ in } k[[1/x]]$$

and then define

$$S(F) = \begin{pmatrix} s_1 & s_2 & \cdots & s_{\mu-1} & s_{\mu} \\ s_2 & s_3 & \cdots & s_{\mu} & s_{\mu-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{\mu-1} & s_{\mu} & \cdots & s_{2\mu-3} & s_{2\mu-2} \\ s_{\mu} & s_{\mu+1} & \cdots & s_{2\mu-2} & s_{2\mu-1} \end{pmatrix} \in S(k)$$

(i.e. $S(F)$ is the Hankel matrix associated to the sequence $s_1, s_2, \ldots, s_{2\mu-1}$).

The other two matrices are defined in terms of a factorization of $f(x)$, so suppose that $L/k$ is a field extension over which the polynomial $f(x)$ factors into linear polynomials, say

$$f(x) = (x - r_1)^{\mu_1} \ldots (x - r_a)^{\mu_a}$$

with $r_i \in L$ and $r_i \neq r_j$ for $i \neq j$. Write the partial fractions decomposition of $g(x)/f(x)$ as

$$g(x)/f(x) = \sum_{i=1}^{a} \sum_{j=1}^{\mu_i} \frac{A_i(j)}{(x - r_i)^j} \text{ with } A_i(j) \in L$$
and then define matrices

\[
\text{New}_i(F) := \begin{pmatrix}
A_i(1) & A_i(2) & \cdots & A_i(\mu_i - 1) & A_i(\mu_i) \\
A_i(2) & A_i(3) & \cdots & A_i(\mu_i) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_i(\mu_i - 1) & A_i(\mu_i) & \cdots & 0 & 0 \\
A_i(\mu_i) & 0 & \cdots & 0 & 0
\end{pmatrix} \in S(L)
\]

and

\[
\text{New}(F) := \begin{pmatrix}
\text{New}_1(F) & 0 & \cdots & 0 & 0 \\
0 & \text{New}_2(F) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \text{New}_{a-1}(F) & 0 \\
0 & 0 & \cdots & 0 & \text{New}_a(F)
\end{pmatrix} \in S(L)
\]

Observe that the matrix \(\text{New}_i(F)\) has entries in the extension \(L\), not the field \(k\). However, the matrix (or rather the \(L\)-vector space with symmetric bilinear form it defines) carries natural descent data (in the sense of e.g. [KW]). When \(L/k\) is Galois, the descent data is defined by assigning to \(\gamma \in \text{Gal}(L/k)\) the permutation matrix \(P_\gamma\) associated to the permutation of the roots induced by \(\gamma\). This defines descent data because

\[
\gamma(\text{New}(F)) = P_\gamma^T \cdot \text{New}(F) \cdot P_\gamma,
\]

where \(\gamma(\text{New}(F))\) is the matrix obtained by applying \(\gamma\) to the coefficients (which permutes the blocks \(\text{New}_i(F)\) as \(\gamma\) permutes the roots \(r_i\)). When \(L/k\) is not Galois (esp. when \(f(x)\) is an inseparable polynomial), it is more difficult to describe the descent data, so we omit its description, but the descent data corresponds to the descent data \(\text{New}(F)\) inherits from its identification as a Gram matrix of a symmetric bilinear form defined over \(k\) (i.e. the description in Lemma 3).

The fourth and final matrix is defined to be the Hankel matrix

\[
\text{Van}(F) = \begin{pmatrix}
\sigma_1 & \sigma_2 & \cdots & \sigma_{\mu-1} & \sigma_\mu \\
\sigma_2 & \sigma_3 & \cdots & \sigma_\mu & \sigma_{\mu+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{\mu-1} & \sigma_\mu & \cdots & \sigma_{2\mu-3} & \sigma_{2\mu-2} \\
\sigma_\mu & \sigma_{\mu+1} & \cdots & \sigma_{2\mu-2} & \sigma_{2\mu-1}
\end{pmatrix} \in S(k).
\]
associated to the sequence

\[ \sigma_1 = \frac{A_1(1)}{g(r_1)^2} + \cdots + \frac{A_n(1)}{g(r_n)^2}, \]

\[ \sigma_2 = \frac{A_1(1)}{g(r_1)^2} r_1 + \frac{A_1(2)}{g(r_1)^2} + \cdots + \frac{A_n(1)}{g(r_n)^2} r_n + \frac{A_n(2)}{g(r_n)^2} \]

\[ \cdots \]

\[ \sigma_b = \sum_{i=1}^{n} \sum_{j=1}^{\mu_i} \frac{A_i(j)}{g(r_i)^2} \binom{b-1}{j-1} r^{b-i} \]

\[ \cdots \]

\[ \sigma_{2\mu-1} = \sum_{i=1}^{n} \sum_{j=1}^{\mu_i} \frac{A_i(j)}{g(r_i)^2} \binom{2\mu-2}{j-1} r^{2\mu-1-i}. \]

Here \( \binom{b-1}{j-1} \) is the binomial coefficient which is taken to equal 0 when \( b < j \).

Observe that while we have defined the \( \sigma_i \)'s as expressions in the roots of \( f(x) \), i.e. as expressions in the extension field \( L \), the expressions are symmetric in the roots, so they in fact lie in \( k \) by the fundamental theorem of symmetric functions.

We conclude this section with a computation involving \( \text{New}(F) \) for an important class of \( F \)'s. Observe that the coefficient \( A_i(\mu) \) in the partial fractions decomposition (4) is nonzero (because e.g. otherwise \( x - r_i \) would be a common divisor of \( f(x) \) and \( g(x) \)), so \( \text{New}_i(F) \) defines an element \( GW(k) \). We compute this element in the following lemma.

**Lemma 2.** If \( A(1), A(2), \ldots, A(\mu) \in k \) are scalars with \( A(1) \) nonzero, then the nondegenerate symmetric matrix

\[
\begin{pmatrix}
A(1) & A(2) & \cdots & A(\mu-1) & A(\mu) \\
A(2) & A(3) & \cdots & A(\mu) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A(\mu-1) & A(\mu) & \cdots & 0 & 0 \\
A(\mu) & 0 & \cdots & 0 & 0
\end{pmatrix}
\in S(k).
\]

represents the Grothendieck–Witt class

\[ w = \begin{cases} 
\langle A(\mu) \rangle + \frac{\mu-1}{2} \cdot \langle 1, -1 \rangle & \mu \text{ odd;} \\
\frac{\mu}{2} \cdot \langle 1, -1 \rangle & \mu \text{ even.}
\end{cases} \]

**Proof.** Consider the matrix

\[
\begin{pmatrix}
A(1) \cdot t & A(2) \cdot t & \cdots & A(\mu-1) \cdot t & A(\mu) \\
A(2) \cdot t & A(3) \cdot t & \cdots & A(\mu) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A(\mu-1) \cdot t & A(\mu) & \cdots & 0 & 0 \\
A(\mu) & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

The determinant is \( \pm A(1)^\mu \), so the matrix defines a nondegenerate symmetric bilinear \( \beta_k \) form on \( (k[t])^{\oplus \mu} \). Arguing as in [KW], we conclude that any two specializations of
the matrix represent the same element of \( GW(k) \). In particular, the original matrix (the specialization at \( t = 1 \)) represents the same Grothendieck–Witt class as the analogous matrix with \( A(1) = \cdots = A(\mu - 1) = 0 \) (the specialization at \( t = 0 \)). Thus we are reduced to the special case where \( A(1) = \cdots = A(\mu - 1) = 0 \).

To prove the lemma in the special case, write \( e_1, \ldots, e_\mu \) for the standard basis of \( k^{\oplus \mu} \). The pairs \((e_1, e_\mu/A(\mu)), (e_2, e_\mu-1/A(\mu)), \ldots \) all span pairwise orthogonal subspaces each isomorphic to the standard hyperbolic plane \( \mathbb{H} \), showing that the matrix in question represents the Grothendieck–Witt class of \( (A(\mu)) + \mathbb{H} + \cdots + \mathbb{H} \), and it is well-known that \( \mathbb{H} = (1, -1) \) in \( GW(k) \), even in characteristic 2.

\[ \square \]

References. The four matrices that we have defined are the four from Kreîn–Naîmark’s survey [KN81] of work related to Hurwitz’s result. That survey does not explicitly discuss the topological degree, but instead discusses an equivalent invariant, namely the Cauchy index. The local Cauchy index \( \text{ind}_r(f(x)/g(x)) \) of \( f(x)/g(x) \in \text{Frac} \mathbb{R}[x] \) at a pole \( r \in \mathbb{R} \) (i.e. at a zero of \( g \)) is defined to be

\[
\text{ind}_r(f(x)/g(x)) := \begin{cases} +1 & \text{if } f(x)/g(x) \text{ jumps from } -\infty \text{ to } +\infty \text{ at } r; \\ -1 & \text{if } f(x)/g(x) \text{ jumps from } +\infty \text{ to } -\infty \text{ at } r; \\ 0 & \text{otherwise}. \end{cases}
\]

The global Cauchy index \( \text{ind}(f(x)/g(x)) \) is the sum of the local Cauchy indices. The fraction \( f(x)/g(x) \) defines a pointed rational function \( F: P^1_\mathbb{R} \rightarrow P^1_\mathbb{R} \) provided the fraction is in lowest terms with \( \text{deg}(f) > \text{deg}(g) \), and under this assumption, the local topological degree \( \text{deg}_{r}(F) \) of \( F: P^1_\mathbb{R}(\mathbb{R}) \rightarrow P^1_\mathbb{R}(\mathbb{R}) \) at a zero \( r \) of \( f(x) \) equals the local Cauchy index of the reciprocal \( g(x)/f(x) \) and the global topological degree \( \text{deg}(F) \) of \( F \) equals the global Cauchy index of \( g(x)/f(x) \).

The fact that the Cauchy index equals the signature of \( \text{Béz}(F) \) can be found in many textbooks that explain how to analyze the zeros of a real polynomial using matrix methods. Gantmacher, Kreîn–Naîmark, and Rahman–Schmeisser attribute this result to Hurwitz but also note that important partial results appear in earlier work of Hermite and Sylvester ([Gan64, footnote 25], [KN81, page 280–281], and [RS02, page 355]). Kreîn–Naîmark’s survey [KN81] provides an extensive survey of the nineteenth century results related to Hurwitz’s result.

Most modern textbooks which discuss Hurwitz’s result define \( \text{Béz}(F) \) and \( S(F) \); see e.g. [LT85, Section 13.3, Proposition 1], [RS02, Definition 10.6.2, Lemma 10.6.6], [BPR06, Notation 9.14, 9.16], [Dym13, page 488, Chapter 21.1]. An exception is Gantmacher’s textbook which defines \( S(F) \) ([Gan64, Equation (55)]) but not the Bézout matrix.

The matrices \( \text{New}(F) \) and \( \text{Van}(F) \) appear less frequently in the literature. Both matrices are defined by Kreîn–Naîmark only under the assumption that \( f(x) \) has distinct roots (\( \text{New}(F) \) in [KN81, Equation (9)]; \( \text{Van}(F) \) in [KN81, Section 5]), although they indicate that the definitions can be extended. The matrices are not explicitly defined in the other texts referenced, although \( \text{New}(F) \) appears, sometimes implicitly, in many of the proofs of
Hurwitz’s result. The notation \(\text{New}(F)\) and \(\text{Van}(F)\) is original to this paper. The notation \(\text{New}(F)\) was chosen to suggest that \(\text{New}(F)\) is related to the Newton basis (i.e. Lemma 3), while the notation \(\text{Van}(F)\) was chosen to suggest that the symmetric matrix is obtained by a change-of-basis matrix related to the Vandermonde matrix.

2. THE RESIDUE PAIRING

Here we recall Scheja–Storch’s construction of the residue pairing \(\beta\) and identify the matrices from Section 1 with Gram matrices for \(\beta\). We then deduce Corollary 5, which is a collection of matrix identities relating the matrices from the previous section to each other. When \(g = 1\), the identification of \(\text{Béz}(F)\) as a Gram matrix was done by Scheja–Storch, and the proof we give here follows along similar lines.

The residue pairing we consider is a distinguished bilinear form associated to a regular sequence in a polynomial ring. The pairing can be constructed as an application of coherent duality, but here we do not use that theory and instead use an explicit construction of Scheja–Storch. They define quite generally, for a ring \(R\) and a length \(n\) regular sequence \(f_1, \ldots, f_n \in R[x_1, \ldots, x_n]\), an \(R\)-linear map

\[
\eta: R[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \to R
\]

we call the residue functional and show that

\[
\beta(b_1, b_2) := \eta(b_1 \cdot b_2)
\]

defines an \(R\)-bilinear nondegenerate symmetric form on the quotient algebra that we call the residue pairing [SS75, page 182]. We recall a characterization of \(\beta\) below at the beginning of the proof of Lemma 3.

With only notational changes, Scheja–Storch’s construction applies to any localization of \(R[x_1, \ldots, x_n]\). The case of importance for this paper is the application of their construction to the algebra

\[
Q(F) := \frac{k[x, 1/g(x)]}{f(x)/g(x)}.
\]

of \(F\). The algebra \(Q(F)\) is closely related to a local algebra \(Q_0(h_0)\) that was studied by Eisenbud, Levine, Khimshiashvili, and others, and we recall the definition of \(Q_0(h_0)\) and its relation to \(Q(F)\) in Section 4.

The algebra \(Q(F)\) admits several natural bases. To define these bases, write

\[
f(x) = x^\mu + a_1 x^{\mu-1} + a_2 x^{\mu-2} + \cdots + a_\mu.
\]

The monomial basis is

\[
1/g(x), x^{1/g(x)}, \ldots, x^{\mu-1}/g(x),
\]
and the **Horner basis** is

\[ x^{m-1}/g(x) + a_1 x^{m-2}/g(x) + \cdots + a_{m-1}/g(x), \]
\[ x^{m-2}/g(x) + a_1 x^{m-3}/g(x) + \cdots + a_{m-2}/g(x), \]
\[ \vdots \]
\[ x/g(x) + a_1/g(x), \]
\[ 1/g(x). \]

When \( f(x) \) factors into linear polynomials, the Chinese Remainder Theorem implies that

\[
\frac{f(x)}{(x - r_1) g(x)} = \frac{1}{g(r_1)} \cdot v_1(x) + \frac{r_1}{g(r_1)} \cdot v_2(x) + \frac{r_1^2}{g(r_1)} \cdot v_3(x) + \cdots + \frac{r_1^{m-1}}{g(r_1)} \cdot v_{m}(x),
\]

\[
\frac{f(x)}{(x - r_1)^2 g(x)} = \frac{1}{g(r_1)} \cdot v_2(x) + 2 \frac{r_1}{g(r_1)} \cdot v_3(x) + \cdots + \frac{(\mu - 1) r_1^{m-2}}{g(r_1)} \cdot v_{m}(x),
\]

\[ \vdots \]

\[
\frac{f(x)}{(x - r_k)^i g(x)} = \sum_{i=1}^{\mu} \left( \begin{array}{c} i - 1 \\ j - 1 \end{array} \right) r_k^{i-j} \frac{1}{g(r_k)} \cdot v_{i}(x),
\]

\[ \vdots \]

\[
\frac{f(x)}{(x - r_n)^{\mu_n} g(x)} = \frac{1}{g(r_n)} \cdot v_{\mu_n}(x) + \cdots + \left( \frac{\mu - 1}{\mu_n - 1} \right) r_n^{\mu - \mu_n} \frac{1}{g(r_n)} \cdot v_{\mu}.
\]

(The basis \( v_1(x), \ldots, v_{\mu}(x) \) exists since the determinant of the confluent Vandermonde matrix is well-known to be \( \pm \prod_{i<j} (r_i - r_j)^{\mu_i \mu_j} \), a nonzero quantity.)

The following lemma explains the relation of \( \beta \) to the matrices:

**Lemma 3.** Each of the matrices \( \text{Béz}(F), \text{S}(F), \text{New}(F), \text{and Van}(F) \) is a Gram matrices for \( \beta \), namely the Gram matrix with respect to the basis dual to the basis in Table 1.

**Table 1. Gram Matrices**

<table>
<thead>
<tr>
<th>Gram Matrix</th>
<th>Dual Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Béz(F)</td>
<td>Monomial basis</td>
</tr>
<tr>
<td>S(F)</td>
<td>Horner basis</td>
</tr>
<tr>
<td>New(F)</td>
<td>Newton basis</td>
</tr>
<tr>
<td>Van(F)</td>
<td>Vandermonde basis</td>
</tr>
</tbody>
</table>
Proof. We begin by proving the claim about the Bézout matrix. If \((1/g(x))^*, \ldots, (x^{\mu-1}/g(x))^*\) is the appropriate dual basis, then we need to prove that
\[
\beta((x^{i-1}/g(x))^*, (x^{j-1}/g(x))^*) = b_{i,j}.
\]
Scheja–Storch prove this result when \(g(x) = 1\) [SS75, page 183, Paragraph 4], and we can extend their result to the general case using a description of \(\beta\) that we now recall.

Let \(\tilde{\Delta} \in k[x, 1/g(x)] \otimes_k k[x, 1/g(x)]\) satisfy
\[
f/g \otimes 1 - 1 \otimes f = (x \otimes 1 - 1 \otimes x) \cdot \tilde{\Delta}
\]
or equivalently
\[
f \otimes g - g \otimes f = (x \otimes 1 - 1 \otimes x) \cdot (g \otimes g \cdot \tilde{\Delta})
\]
and \(\Delta \in Q(F) \otimes_k Q(F)\) be its image. The isomorphism
\[
k[x] \otimes_k k[x] \cong k[x, y],
\]
\[
p(x) \otimes q(x) \mapsto p(x)q(y)
\]
transforms equation (6) into (3) and hence \(\tilde{\Delta}\) into \(\sum b_{i,j}x^{i-1}y^{j-1}\).

On [SS75, page 182, bottom] Scheja–Storch show that if \(v_1, \ldots, v_\mu\) and \(v_1^*, \ldots, v_\mu^*\) are bases dual with respect to \(\beta\), then
\[
\Delta = v_1 \otimes v_1^* + \cdots + v_\mu \otimes v_\mu^*.
\]
or equivalently
\[
\frac{1}{g(x)g(y)} \cdot \sum b_{i,j}x^{i-1}y^{j-1} = v_1(x)v_1^*(y) + \cdots v_\mu(x)v_\mu^*(y).
\]
Since
\[
\sum_{i=1}^\mu \frac{x^{i-1}}{g(x)} \cdot \left(\frac{y^{i-1}}{g(y)}\right)^* = \frac{1}{g(x)g(y)} \cdot \sum_{i,j=1}^\mu b_{i,j}x^{i-1}y^{j-1} = \sum_{i=1}^\mu \frac{x^{i-1}}{g(x)} \cdot \left(\sum_{j=1}^\mu b_{i,j} \frac{y^{j-1}}{g(y)}\right),
\]
we can conclude that
\[
\left(\frac{x^{i-1}}{g(x)}\right)^* = \sum_{j=1}^\mu b_{i,j} \frac{x^{j-1}}{g(x)}.
\]
In particular,
\[
\beta\left(\left(\frac{x^{i-1}}{g(x)}\right)^*, \left(\frac{x^{j-1}}{g(x)}\right)^*\right) = \sum_{k=1}^\mu b_{i,k} \cdot \beta\left(\frac{x^{k-1}}{g(x)}, \left(\frac{x^{j-1}}{g(x)}\right)^*\right) = b_{i,j}.
\]
The claims about $S(F)$ and $New(F)$ are proven in a similar manner. For $S(F)$, we need to establish the identity

\[
\frac{f(x)/g(x) - f(y)/g(y)}{x - y} = \sum_{i,j=1}^{\mu} s_{i+j-1} \frac{x^{\mu-i} + a_1 x^{\mu-i-1} + \cdots + a_{\mu-i} y^{\mu-j} + a_1 y^{\mu-j-1} + \cdots + a_{\mu-j}}{g(x) g(y)},
\]

and this is deduced using elementary algebra from the identity

\[
\frac{f(x)/g(x) - f(y)/g(y)}{x - y} = \frac{g(y)/f(y) - g(x)/f(x)}{x - y} \cdot \frac{f(x)f(y)}{g(x)g(y)}.
\]

(7)

We deduce the claim about $New(F)$ by using the identity

\[
\frac{g(y)}{f(y)} - \frac{g(x)}{f(x)} = \sum_{i=1}^{a} \sum_{j=1}^{\mu_i} A_i(j) (y - r_i)^{-j} - \frac{A_i(j)}{(x - r_i)^j} = (x - y) \cdot \sum_{i=1}^{a} \sum_{j=1}^{\mu_i} A_i(j) \cdot ((x - r_i)^{-1}(y - r_i)^{-j} + \cdots + (x - r_i)^{-j}(y - r_i)^{-1}).
\]

Finally, by definition, the change-of-basis matrix relating the Vandermonde basis to the Newton basis is the modified confluent Vandermonde matrix, and a computation shows that this change-of-basis transforms $New(F)$ into $Van(F)$; see Corollary 5 for details.

From Lemma 3, we immediately deduce that the matrices $Béz(F), S(F), New(F)$, and $Van(F)$ are related by change-of-basis matrices. We record this in the following definition and lemma.

**Definition 4.** Define the matrix $L$ by

\[
L = \begin{pmatrix}
a_{\mu-1} & a_{\mu-2} & \cdots & a_1 & 1 \\
a_{\mu-2} & a_{\mu-3} & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_1 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

For $i = 1, \ldots, n$, define $M_i$ to be the $\mu$-by-$\mu_i$ matrix with entries $m_i(j_1, j_2)$ recursively defined by

\[
m_i(j_1, j_2) = \begin{cases}
1 & \text{if } (j_1, j_2) = (1, \mu); \\
r_1 \cdot m_i(j_1 + 1, j_2) + a_{\mu-j_2} & \text{if } j_1 = 1, j_2 \neq \mu; \\
0 & \text{if } j_1 \neq 1, j_2 = \mu; \\
r_1 \cdot m_i(j_1 + 1, j_2) + m_i(j_1 + 1, j_2 - 1) & \text{otherwise},
\end{cases}
\]

(8)

(So $m_i(j_1, j_2)$ is defined in terms of the entries of the submatrix

\[
\begin{pmatrix}
m_i(j_1, j_2 - 1) & m_i(j_1, j_2) \\
m_i(j_1 + 1, j_2 - 1) & m_i(j_1 + 1, j_2)
\end{pmatrix}.
\]

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Define $M$ to be the 1-by-$n$ block matrix
\[ M := \begin{pmatrix} M_1 & M_2 & \cdots & M_n \end{pmatrix}. \]

For $k = 1, \ldots, n$, define $V_k$ to be the $\mu_k \times \mu$ modified confluent Vandermonde matrix $V_k := \left( \binom{i+j-1}{j-1} r_k^{i-j}/g(r_k) \right)$ and define $N_0$ to be the inverse of the associated 1-by-$n$ block matrix
\[ N_0^{-1} := \begin{pmatrix} V_1 & V_2 & \cdots & V_n \end{pmatrix}. \]

Define
\[ N := M \cdot N_0. \]

**Corollary 5.** We have
\[ B\acute{e}z(F) = L \cdot S(F) \cdot L^T = M \cdot \text{New}(F) \cdot M^T = N \cdot \text{Van}(F) \cdot N^T. \]

**Proof.** This follows from Lemma 3 and the rule describing how a Gram matrix transforms under a change of basis. Recall quite generally that if $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases for a vector space $V$ that are related by $w_j = \sum b_{i,j} v_i$, then the two associated Gram matrices for a given bilinear form $\beta$ are related by
\[
\begin{pmatrix}
    b_{1,1} & \cdots & b_{n,1} \\
    \vdots & \ddots & \vdots \\
    b_{1,n} & \cdots & b_{n,n}
\end{pmatrix}^T \begin{pmatrix}
    \beta(v_1, v_1) & \cdots & \beta(v_n, v_1) \\
    \vdots & \ddots & \vdots \\
    \beta(v_1, v_n) & \cdots & \beta(v_n, v_n)
\end{pmatrix} \begin{pmatrix}
    b_{1,1} & \cdots & b_{n,1} \\
    \vdots & \ddots & \vdots \\
    b_{1,n} & \cdots & b_{n,n}
\end{pmatrix} = 
\begin{pmatrix}
    \beta(w_1, w_1) & \cdots & \beta(w_n, w_1) \\
    \vdots & \ddots & \vdots \\
    \beta(w_1, w_n) & \cdots & \beta(w_n, w_n)
\end{pmatrix}.
\]

In the present setting, $V = Q(F)$, $w_1, \ldots, w_n$ is the dual of the monomial basis and $v_1, \ldots, v_n$ is the dual to one of the other bases from Table 1. Thus if we write the basis $v_1^*, \ldots, v_n^*$ in terms of the monomial basis $w_1^*, \ldots, w_n^*$ as $v_j^* = \sum a_{i,j} w_i^*$, then $(b_{i,j}) = (a_{i,j})^T$. The claim is therefor that the columns of $L$, $M$, and $N$ respectively express Horner, Newton, and Vandermonde bases in terms of the monomial basis.

The claim about the Horner basis is immediate from the definition. To verify the claim about the Newton basis, observe that the identity $(x - r_1) \cdot f(x)/(x - r_1)^{i+1} = f(x)/(x - r_1)^i$ implies that the coefficients of $f(x)/(x - r_1), f(x)/(x - r_1)^2, \ldots$ satisfy the recursion (8). Finally, given the validity of the claim concerning the Newton basis, to verify the claim about the Vandermonde basis, it is enough to show that the columns of $N$ express the Vandermonde basis in terms of the Newton basis, and this holds by definition of the Vandermonde basis. \[ \square \]
3. The Main Theorem

In this section we use the results of the previous two sections to prove the Main Theorem. We also discuss the barrier to extending the arguments of this paper to a proof that $\text{B}ez(F)$ represents the unstable homotopy class of $F$.

An important tool we use in proving the Main Theorem is a formula relating $\deg^A_1(F)$ to a sum of local $A^1$-degrees. Recall that the local $A^1$-degree is defined in terms of Morel’s degree homomorphism in the natural manner. Namely, suppose that $\overline{x} \in P^1_k(k)$ is a $k$-rational point with $\overline{y} := F(\overline{x})$. The Purity theorem [MV99, Section 3, Proposition 2.17 and Theorem 2.23] gives a canonical homotopy equivalence between $P^1_k$ and the quotient $P^1_k - \{\overline{x}\}$, and more generally with $\frac{U}{U - \{\overline{x}\}}$ for $U \subset P^1_k$ a Zariski open subset containing $\overline{x}$. Thus picking a Zariski neighborhood $U \subset A^1_k$ that excludes all points of $F^{-1}(\overline{y})$ other than $\overline{x}$, we can define the local $A^1$-degree $\deg_A^1(\overline{y})$ to be the $A^1$-degree of the induced morphism

$$P^1_k \cong \frac{U}{U - \{\overline{x}\}} \rightarrow \frac{P^1_k}{P^1_k - \{\overline{y}\}} \cong P^1_k.$$ 

The local $A^1$-degree is related to the $A^1$-degree as follows. If $F^{-1}(\overline{y})$ is supported on a collection of $k$-rational points $\{\overline{x}_1, \ldots, \overline{x}_n\}$, then

$$\tag{9} \deg^A_1(F) = \deg_{\overline{x}_1}^A(\overline{y}) + \cdots + \deg_{\overline{x}_n}^A(\overline{y})$$

by the same proof as in, for instance, [Hat02, Proposition 2.30].

More generally, if $\overline{y} \in P^1_k(k)$ and $L/k$ is a finite field extension that splits the residue field of every point in the support of $F^{-1}(\overline{y})$, then summing over the support $\overline{x}_1, \ldots, \overline{x}_n$ of $L \otimes_k F^{-1}(\overline{y})$, the right-hand side of (9) is an element of $GW(L)$ that is the image of a distinguished element under the natural morphism $GW(k) \rightarrow GW(L)$. The distinguished element is defined as follows. If we pick symmetric bilinear forms $\beta_1, \ldots, \beta_n$ respectively representing the elements $\deg_{\overline{x}_1}^A(F), \ldots, \deg_{\overline{x}_n}^A(F)$, then the direct sum $\beta_1 \oplus \cdots \oplus \beta_n$ carries natural descent data corresponding to the descent data on $L \otimes_k F^{-1}(\overline{y})$, and this bilinear form with descent data represents $\deg^A_1(F)$.

Unfortunately, there does not seem to be a published proof that this bilinear form with descent data represents $\deg^A_1(F)$ in the case that $L$ is an inseparable extension of $k$. The formula is stated in the special case that $f^{-1}(y)$ is a disjoint union of copies of $\text{Spec}(k)$ by Morel in [Mor06, page 1036] and [Mor04, Section 2] and by Levine in [Lev08, page 188] in the special case where $f^{-1}(y)$ is étale, and both authors indicate that the formula holds in greater generality. In the generality we have stated it, this descent result can be deduced using Cazanave’s result and the arguments used below in the proof of the Main Theorem, but it would be desirable to have a proof of the result that does not use Cazanave’s result. The present authors discuss Formula (9), the descent result, and the generalization of these results to a polynomial maps $F: P^n_k \rightarrow P^n_k$ with $n \geq 1$ in greater depth in the companion article [KW].
We now prove the Main Theorem, beginning with a direct proof of theorem in the special case of a power map.

**Lemma 6.** If $F$ is defined by $f(x)/g(x) = x^\mu/1$ with $\mu \geq 1$, then

\[
\deg A^1(F) = \begin{cases} 
\langle (1) + \frac{\mu-1}{2} \cdot (1, -1) \rangle & \mu \text{ odd;} \\
\frac{\mu}{2} \cdot (1, -1) & \mu \text{ even.}
\end{cases}
\]

**Proof.** We prove the result by induction on $\mu$. The result holds for $\mu = 1$ by the construction of $\deg A^1$, so we assume the result holds for $\mu$ and then prove it holds for $\mu + 1$. Consider the auxiliary function $F_0: P^1_k \to P^1_k$ defined by the polynomial $x^{\mu+1} + x^\mu$. We compute $\deg A^1(F_0)$ in two different ways.

First, the expression $x^{\mu+1} + tx^\mu$ defines a naive $A^1$-homotopy from $F_0$ to $x^{\mu+1}$, so

\[
\deg A^1(F_0) = \deg A^1(x^{\mu+1}).
\]

Second, by (9), we have that

\[
\deg A^1(F_0) = \deg A^1_0(F_0) + \deg A^1_{-1}(F_0).
\]

We compute the two local degrees using the finite determinacy result of [KW]. That result shows that if $x^b$ occurs with nonzero coefficient in $f(x)$, then $\deg A^1_0(f) = \deg A^1_0(f + g)$ provided $g \in (x^{b+1})$. By translation, the analogous result holds with $-1$ replacing $0$. It follows that

\[
\deg A^1(F_0) = \deg A^1_0(F_0) + \deg A^1_{-1}(F_0)
\]

\[
= \deg A^1_0(x^\mu) + \deg A^1_{-1}((-1)^{\mu+1}(x + 1))
\]

\[
= \deg A^1(\langle x^\mu \rangle + (-1)^{\mu+1}).
\]

We conclude that $\deg A^1(x^{\mu+1}) = \deg A^1(\langle x^\mu \rangle + (-1)^{\mu+1})$, completing the proof by induction. \hfill \Box

**Proof of Main Theorem.** The result is an immediate consequence of Lemma 3 and Cazanave’s result. If we do not use Cazanave’s result but assume $\deg A^1(F)$ can be computed from local $A^1$-degrees and the descent data on the roots $\{r_1, \ldots, r_n\}$ in the manner described at beginning of this section (recall the assumption is true, but there is a published proof independent of Cazanave’s result only when the roots generate a separable extension of $k$), then an alternative proof of the weaker result that the stable $A^1$-homotopy class of $F$ is represented by $Béz(F)$ is as follows.

It is sufficient to show that $\deg A^1(F)$ is represented by $\text{New}(F)$ with its descent data because $\text{New}(F)$ is $\text{GL}_\mu$-equivalent to $Béz(F)$ by Corollary 5. The special case where $F$ is defined by the fraction $(x-r)^\mu/A$ follows from Lemmas 2 and 6. Indeed, Lemma 6 implies

\[
\deg A^1((x-r)^\mu/A) = \begin{cases} 
\langle A \rangle + \frac{\mu-1}{2} \cdot \langle A, -A \rangle & \mu \text{ odd;} \\
\frac{\mu}{2} \cdot \langle A, -A \rangle & \mu \text{ even}
\end{cases}
\]

since $\deg A^1$ transforms composition of rational functions into multiplication in $GW(k)$ and $\deg A^1(x/A) = \langle A \rangle$ [Caz12, page 523]. By Lemma 2, the class on the right-hand side of (12) is the class represented by $\text{New}(F)$, so we have proven the result when $F = (x-r)^\mu/A$. 


The general case can be deduced from the special case as follows. Fix a finite field extension \( L/k \) in which \( f(x) \) factors into linear polynomials, say as \( f(x) = (x - r_1)^{\mu_1} \cdots (x - r_n)^{\mu_n} \). Both \( \deg A^i(F) \) and the Grothendieck–Witt class represented by \( \text{New}(F) \) equal the element of \( \text{GW}(k) \) determined by

\[
\deg A^i_1(F) + \cdots + \deg A^i_{\mu}(F) \in \text{GW}(L)
\]

together with the natural descent data on \( \{ r_1, \ldots, r_n \} \). Indeed, that this class equals \( \deg A^i(F) \) is the degree formula (9) and the descent result, and that this class equals the class represented \( \text{New}(F) \) follows from Corollary 5, Equation (12) and Lemma 2. \( \square \)

**Corollary 7.** The matrices \( \text{Béz}(F), S(F), \text{New}(F), \text{and } \text{Van}(F) \) represent the stable homotopy class of \( F \). The matrices \( \text{Béz}(F) \) and \( S(F) \) represent unstable homotopy class of \( F \).

**Proof.** The claim about the stable homotopy class follows immediately from the Main Theorem and Corollary 5. The claim about the unstable homotopy class follows as \( \text{det}(L) = \pm 1 \). \( \square \)

**Remark 8.** The matrices \( \text{New}(F) \) and \( \text{Van}(F) \) do not always represent the unstable homotopy class of \( F \). For example, when \( k = \mathbb{C}(r_1, r_2) \) and \( F \) is defined by \( f(x)/g(x) = (x - r_1)(x - r_2)/1 \), the matrix \( M \) has determinant \( \text{det}(M) = r_1 - r_2 \).

Similarly, if \( k = \mathbb{C}(r_1, r_2, b_0, b_1) \) and \( f(x)/g(x) = (x - r_1)(x - r_2)/(b_0x + b_1) \), then \( \text{det}(N) = -(b_0r_1 + b_1)(b_0r_2 + b_1) \).

The proof just given shows, independently of Cazanave’s result, that \( \text{Béz}(F) \) represents the stable \( A^i \)-homotopy class of \( F \). We conclude this section by discussing the obstruction to using a similar argument to show \( \text{Béz}(F) \) represents the unstable homotopy class. The obstruction is clearly demonstrated in the special case where \( f(x) \) has \( \mu \) distinct roots defined over \( k \). In this case, the coefficient \( A_i(\mu) \) from (4) equals \( g(r_i)/f'(r_i) \), and so the proof of the Main Theorem shows that \( \deg A^i(F) \) can be expressed as

\[
\deg A^i(F) = \deg r_1^{\mu_1}(F) + \cdots + \deg r_n^{\mu_n}(F)
\]

\[
= (g(r_1)/f'(r_1), \ldots, g(r_\mu)/f'(r_\mu)).
\]

The unstable homotopy class of \( F \) is determined by \( \deg A^i(F) \) together with a scalar \( d(F) \) that represents the discriminant of \( \deg A^i(F) \), and it is tempting to speculate that \( d(F) = g(r_1)/f'(r_1) \cdots g(r_\mu)/f'(r_\mu) \), but this is not the case. Indeed, Cazanave’s work shows

\[
d(F) = \text{det}(\text{Béz}(F))
\]

\[
= (-1)^{(n-1)/2} \text{Res}(f, g),
\]

so \( d(F) \) is related to the derivatives of \( F \) by

\[
d(F) = \text{Disc}(f) \cdot g(r_1)/f'(r_1) \cdots g(r_\mu)/f'(r_\mu).
\]

Equation (13) shows that, unlike \( \deg A^i(F) \), \( d(F) \) is not determined by the derivatives of \( f(x)/g(x) \) at the roots: For example both \( f_1(x)/g_1(x) = x^2 - x \) and \( f_2(x)/g_2(x) = (x^2 - 1)/2 \).
have two simple zeros at which the values of the derivative are $+1$ and $-1$ respectively, but the unstable homotopy classes of the associated morphisms $F_1, F_2: P^1_k \to P^1_k$ are not equal since $d(F_1) = -1$ but $d(F_2) = -4$. To extend the proof of the weakened form of the Main Theorem to a proof that the unstable homotopy class of $F$ is determined by $B\acute{e}z(F)$, it would be sufficient to prove Equation (13), and its generalization to the case where $f$ has repeated roots, without using Cazanave’s result.

4. CONNECTION WITH WORK OF EISENBUD–LEVINE AND KHMISHIAHVILI

The Main Theorem of this paper is related to the signature formula of Eisenbud–Levine and Khimshiashvili. In this section we recall the signature formula, describe its relation to the present work, and announce the main result of [KW].

The signature formula computes the local topological degree of the germ $h_0: (R^n, 0) \to (R^n, 0)$ of a $C^\infty$-function as the signature of a bilinear form. The formula applies when $h_0$ has the property that the local algebra

$$Q_0(h_0) := C^\infty_0(R^n)/(h_0).$$

has finite length. Here $C^\infty_0(R^n)$ is the ring of germs of smooth real-valued functions on $R^n$ based at 0 and $(h_0)$ is the ideal generated by the components of $h_0$.

When the length condition holds, $Q_0(h_0)$ carries a distinguished symmetric bilinear form $\beta_0$, essentially by the construction of Scheja–Storch. The signature formula states

$$\deg_0(h_0) = \text{signature of } \beta_0,$$

where $\deg_0(h_0)$ is the topological local degree $\deg_0(h_0) \in Z$ of $h_0$ ([EL77, Theorem 1.2], [Khi77]; see [Khi01] and [AGZV12, Chapter 5] for recent expositions).

In [BCRS96], Becker–Cardinal–Roy–Szafraniec observed that the classical result of Hurwitz discussed in the introduction is closely related to the signature theorem in the special case where $n = 1$ and $h_0$ is the germ of a pointed rational function. If $h_0$ is the germ of a real pointed rational function $F: P^1_R \to P^1_R$ that vanishes at the origin, then the natural map $R[x, 1/g(x)] \to C^\infty_0(R)$ induces an isomorphism

$$Q_0(F) \to Q_0(h_0)$$

between $Q_0(h_0)$ and the localization $Q_0(F)$ of $Q(F)$ at the maximal ideal $(x)$ of the origin, by the finiteness of $Q_0(h_0)$. Moreover, since $Q(F)$ is an artin algebra, $Q_0(F)$ is a direct summand of $Q(F)$, and the restriction of $\beta$ to this summand is $\beta_0$. Thus the signature theorem applied to the $h_0$ in question asserts

$$\deg_0(F) = \text{signature of } \beta_0.$$

Equation (16) is real realization of

$$\deg_0^{A^1}(F) = w(\beta_0),$$

and in proving the Main Theorem, we proved that Equation (17) holds when $k$ is an arbitrary field.
The equality $\deg_{A^1} (F) = w(\beta_0)$ answers a question of Eisenbud for polynomial maps of 1 variable. Eisenbud, in a survey article on his work with Levine, observed that $w(\beta_0)$ is defined for a polynomial map $f: A^n_k \to A^n_k$ with an isolated zero at the origin when $k$ is an arbitrary field of odd characteristic. He then proposed $w(\beta_0)$ as the definition of the degree, and asked if this degree has an interpretation or usefulness, say in cohomology theory [Eis78, Some remaining questions (3)].

In this paper we have shown that $w(\beta_0)$ is the local degree in $A^1$-homotopy theory when $f$ is a polynomial map of 1 variable. In the companion paper [KW], the present authors answer Eisenbud’s question in full generality: For a polynomial map $f: A^n_k \to A^n_k$ with an isolated zero at the origin, we have

$$\deg_{A^1} (f) = w(\beta_0),$$

as stated in Theorem 1.

Finally, we wish to point out a subtle but significant point about the Main Theorem and its connection to the work of Eisenbud–Levine and Khimshiashvili. While Scheja–Storch construct an explicit symmetric bilinear form $\beta$, Eisenbud–Levine emphasize working with the stable isomorphism class $w(\beta)$ of that form, and observe that there are other natural representatives of that class: For any $k$-linear function $\eta': Q(F) \to k$ that coincides with the residue functional on the socle, the form $\beta'(b_1, b_2) := \eta'(b_1 \cdot b_2)$ is isomorphic to $\beta$ [EL77, Proposition 3.5]. However, when $n = 1$ interesting topological information is lost in passing from $\beta$ to $w(\beta)$. The isomorphism class $w(\beta)$ determines the $A^1$-degree, and the $A^1$-degree determines the unstable pointed homotopy class for $n \geq 2$ but not for $n = 1$. For $n = 1$, the unstable pointed homotopy class is determined by the $A^1$-degree together with a distinguished scalar representing the discriminant. The scalar, and hence the homotopy class, can be recovered from $\beta$ because the scalar is the determinant of the Gram matrix of $\beta$ with respect to a natural basis, the basis dual to $1/g(x), x/g(x), x^2/g(x), \ldots, x^{n-1}/g(x)$.

5. ACKNOWLEDGEMENTS

TO BE ADDED AFTER THE REFEREE PROCESS.

6. OLD INTRO

Uncomment to get old intro.

7. TO DO

Check (13). Is the sign correct? I check this for examples and the signs worked out correctly.

Check references in Cazanave. [Checked! — Jesse]
Should we call $F$ a rational function or a rational map?

Modify definition of $\text{Van}(F)$ so it is SL-equivalent to Bezout?

Give general definition of $\text{Van}(F)$?

The general definition of $\sigma_n$ is as follows. Redefine index $A_i(a)$ to be $A_i(\mu_i - a)$. Then the coefficients $\sigma_n$ for $n = 0, 1, 2, \ldots$ are defined by

$$\sigma_n = \sum_{i=1}^{n} \sum_{a=1}^{\mu_i} A_i(a) \binom{n-1}{a-1} r^{n-a}.$$  

The general definition should have $\sigma_a = \sum_{i=1}^{n} \sum_{\alpha=1}^{\mu_i} A_i(\alpha) r^{a-\alpha} \binom{a-2}{\alpha-2}$. Maybe $A(\mu_i - 1)$.

The general definition of the Jacobi basis should satisfy $e_j = \sum_{i=1}^{\mu_i} \binom{i-1}{j-1} r^{i-j} f_i$.

Change Jacobi basis to Vandermonde basis? (Jacobi is overloaded.)

Add references for the definitions of matrices? The names of the Bezout matrix and $S(F)$ are totally standard, but I made up the other two definitions.

REFERENCES


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