AN ELEMENTARY PROOF THAT THE ALGEBRAIC HOMOTOPY CLASS OF A RATIONAL FUNCTION IS THE SCHEJA–STORCH FORM

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Abstract. C. Cazanave has identified the naive algebraic homotopy class of a rational function of one variable with an explicit nondegenerate symmetric bilinear form. Here we identify Cazanave’s form with a bilinear form constructed by Scheja and Storch in commutative algebra and we announce related work identifying the local A1-degree of a polynomial map in several variables that has an isolated zero at the origin with the Eisenbud–Levine–Khimshiashvili form. The proof is elementary in the sense that it uses no A1-homotopy theory beyond Cazanave’s explicit description.

In the papers [Caz08, Caz12] Cazanave described the set $\{P_1^1, P_1^k\}^N$ of pointed naive homotopy classes of pointed rational functions in one variable over a field $k$ as follows: Given a pointed rational function $F: P_1^k \to P_1^k$, Cazanave defines a symmetric matrix $\text{Béz}(F)$ (the Bézout matrix) and proves that $F \mapsto \text{Béz}(F)$ defines a bijection between the set of naive homotopy classes and the set of pairs consisting of a stable isomorphism class of a nondegenerate symmetric bilinear form $w$ and a scalar $d \in k^\times$ that represents the discriminant $\text{disc}(w) \in k^\times/(k^\times)^2$ [Caz12, Proposition 3.9]. This second set is identified with the $k$-points of the naive connected components $\pi_0^N(S)$ of the scheme $S$ of nondegenerate symmetric matrices [Caz12, Theorem 3.6], so the bijection can be written as

$$[P_1^1, P_1^k]_N \cong \pi_0^N(S).$$

Cazanave’s result compares well with a celebrated construction of Morel. Morel constructed a degree homomorphism

$$\text{deg}^{A^1}: [(P_1^1)^{\wedge n}, (P_1^k)^{\wedge n}] \to GW(k)$$

from the set of pointed $A^1$-homotopy classes of pointed endomorphisms of $(P_1^k)^{\wedge n}$ to the Grothendieck–Witt group of nondegenerate symmetric bilinear forms, and this homomorphism identifies $[P_1^k, P_1^k]$ with the set of pairs consisting of an element $w \in GW(k)$ and a scalar representing $\text{disc}(w)$ in a manner compatible with Cazanave’s description of $[P_1^1, P_1^1]_N$ ([Mor12, Theorem 7.36] for $k$ perfect; see [Mor12, Remark 1.17] and [Hoy14, Footnote 1] for $k$ imperfect).

In this paper, we show that $\text{Béz}(F)$ has an interpretation in the theory of duality for 0-dimensional rings. To simplify ideas, assume that $F$ is defined by a monic polynomial $f(x) \in k[x]$. Scheja–Storch have then constructed a symmetric bilinear form on the 0-dimensional algebra $Q(F) := k[x]/f(x)$ of the function. We prove:

Main Theorem. Assume $F: P_1^k \to P_1^k$ is defined by $f(x) \in k[x]$. Then $\text{Béz}(F)$ is the Gram matrix of the Scheja–Storch form with respect to the basis dual to the basis of monomials $1, x, x^2, \ldots$
We deduce this result from Theorem 1 of Section 1, a more general result that applies to an arbitrary pointed rational function.

The Main Theorem is related to the beautiful formula of Eisenbud–Levine and Khimshiashvili that the local degree (or index) of a real polynomial function is the signature of the symmetric bilinear form of Scheja–Storch. In Section 2 we discuss that work and its relation to the Main Theorem, and we announce the more general result that the local $A^1$-degree of a polynomial map of $n$ variables is the stable isomorphism class of the Scheja–Storch form.

CONVENTIONS

A pointed rational function is a nonconstant $k$-morphism $F: \mathbb{P}^1_k \to \mathbb{P}^1_k$ satisfying $F(\infty) = \infty$. Every such morphism can be represented by a fraction $f(x)/g(x) \in \text{Frac} \ k[x]$ with $f$ monic and $\deg f < \deg g$.

Given two pairs $(V_1, \beta_1), (V_2, \beta_2)$ each consisting of a $k$-vector space $V_i$ and a nondegenerate bilinear pairing $\beta_i$, we say that $(V_1, \beta_1)$ is stably isomorphic to $(V_2, \beta_2)$ if there exists a third such pair $(V_3, \beta_3)$ such that $(V_1 \oplus V_3, \beta_1 \oplus \beta_3)$ is isomorphic to $(V_2 \oplus V_3, \beta_2 \oplus \beta_3)$.

The Gram matrix of a symmetric bilinear form $\beta$ on a finite dimensional $k$-vector space $V$ with respect to a basis $e_1, e_2, \ldots, e_\mu$ is the matrix $(\beta(e_i, e_j))$.

1. THE MAIN THEOREM

In this section we prove the Main Theorem, which relates Cazanave’s description of the pointed naive homotopy class of a pointed rational function to the Scheja–Storch bilinear form. In [Caz12] Cazanave constructs, for a given pointed rational function $F: \mathbb{P}^1_k \to \mathbb{P}^1_k$, an explicit nondegenerate symmetric matrix $\text{Béz}(F)$ defined as follows: If $F$ is defined by a fraction $f(x)/g(x) \in \text{Frac} \ k[x]$ written in lowest terms with $f(x)$ monic, then Cazanave writes

$$ f(x)g(y) - f(y)g(x) = (x - y) \cdot \left( \sum_{i,j=0}^{\mu-1} c_{i+1,j+1} x^i y^j \right) \quad \text{in} \ k[x, y], $$

with $\mu$ the degree of $f$ and defines

$$ \text{Béz}(F) := (c_{i,j}). $$

This last expression is a nondegenerate symmetric matrix, and we compare it to the following construction of Scheja–Storch. Quite generally for a ring $A$ and a length $n$ regular sequence $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$, Scheja–Storch define an $A$-linear map

$$ \eta: A[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \to A $$

and show that

$$ \beta(b_1, b_2) := \eta(b_1 \cdot b_2) $$
defines an $A$-bilinear nondegenerate symmetric form on the quotient algebra [SS75, page 182]. We omit the definition of $\eta$ because it is long and we do not make direct use of it. We do, however, recall in the proof of Theorem 1 below an alternative characterization of the form.

The construction of Scheja–Storch applies to the algebra of $F$

$$Q(F) := \frac{k[x, 1/g(x)]}{f(x)/g(x)},$$

and we prove:

**Theorem 1.** The matrix $Béz(F)$ is the Gram matrix of the Scheja–Storch form of $Q(F)$ with respect to the basis dual to the basis $1/g(x), x/g(x), x^2/g(x), \ldots, x^{\mu-1}/g(x)$.

**Proof.** Say $(1/g(x))^*, \ldots, (x^{\mu-1}/g(x))^*$ is the relevant dual basis. We need to prove

$$\beta((x^i-1/g(x))^*, (x^{j-1}/g(x))^*) = c_{ij}.$$ 

We do so using a description of $\beta$ that we now recall.

Let $\tilde{\Delta} \in k[x, 1/g(x)] \otimes_k k[x, 1/g(x)]$ satisfy

$$f/g \otimes 1 - 1 \otimes f/g = (x \otimes 1 - 1 \otimes x) \cdot \tilde{\Delta},$$

or equivalently

(5)$$f \otimes g - g \otimes f = (x \otimes 1 - 1 \otimes x) \cdot (g \otimes g \cdot \tilde{\Delta}).$$

and $\Delta \in Q(F) \otimes_k Q(F)$ be its image. On [SS75, page 182, bottom] Scheja–Storch show that if $b_1, \ldots, b_\mu$ and $(b_1)^*, \ldots, (b_\mu)^*$ are bases dual with respect to $\beta$, then

$$\Delta = b_1 \otimes (b_1)^* + \cdots + b_\mu \otimes (b_\mu)^*.$$ 

The isomorphism

$$k[x] \otimes_k k[x] \cong k[x, y],$$

$$p(x) \otimes q(x) \mapsto p(x)q(y)$$

transforms equation (5) into (3) and hence $\tilde{\Delta}$ into $\frac{1}{g(x)g(y)} \cdot \sum c_{ij}x^{i-1}y^{j-1}$. Since

$$\sum_{i=1}^{\mu} \frac{x^{i-1}}{g(x)} \cdot \left(\frac{y^{i-1}}{g(y)}\right)^* = \frac{1}{g(x)g(y)} \cdot \sum_{i,j=1}^{\mu} c_{ij}x^{i-1}y^{j-1} = \sum_{i=1}^{\mu} \frac{x^{i-1}}{g(x)} \cdot \left(\sum_{j=1}^{\mu} c_{ij} \frac{y^{j-1}}{g(y)}\right),$$

we can conclude that

$$\left(\frac{x^{i-1}}{g(x)}\right)^* = \sum_{j=1}^{\mu} c_{ij} \frac{x^{j-1}}{g(x)}.$$
In particular,

\[
\beta \left( \left( \frac{x^{i-1}}{g(x)} \right)^* , \left( \frac{x^{j-1}}{g(x)} \right)^* \right) = \sum_{k=1}^{\mu} c_{i,k} \cdot \beta \left( \frac{x^{k-1}}{g(x)} , \left( \frac{x^{j-1}}{g(x)} \right)^* \right) = c_{i,j}.
\]

We deduce the Main Theorem as the special case of Theorem 1 when \( g(x) = 1 \).

2. Connection with work of Eisenbud–Levine and Khimshiashvili

The Main Theorem of this paper is related to a result of Eisenbud–Levine and Khimshiashvili that we now recall. When \( k = \mathbb{R} \) (the real numbers) and \( f: \mathbb{A}^n_{\mathbb{R}} \to \mathbb{A}^n_{\mathbb{R}} \) is a polynomial map with an isolated zero at the origin, the construction of Scheja–Storch produces a symmetric bilinear form \( \beta_0 \) on the localization of the algebra \( Q(f) \) of \( f \) at the origin. Eisenbud–Levine and Khimshiashvili prove that the signature of \( \beta_0 \) is the topological local degree \( \deg_0(f) \in \mathbb{Z} \) of \( f \) at the origin ([EL77, Theorem 1.2], [Khi77]; see [AGZV12, Chapter 5] and [Khi01] for recent expositions). An earlier result of Palamodov implies the rank of \( \beta_0 \) is the topological local degree of the complexification \( \deg_0(f_{\mathbb{C}}) \in \mathbb{Z} \) [Pal67, Corollary 4]. The signature and rank of \( \beta_0 \) determine its stable isomorphism class (or Grothendieck–Witt class) \( w(\beta_0) \), which we call the Eisenbud–Khimshiashvili–Levine class to distinguish it from the bilinear form of Scheja–Storch. When \( n = 1 \) and \( f \) is the restriction of a pointed rational function \( F: \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}} \), the results just discussed imply that \( w(\beta_0) \) is a local contribution to the \( \mathbb{A}^1 \)-Brouwer degree \( \deg_{\mathbb{A}^1}(F) \) of \( F \). The Main Theorem shows that this fact remains true when \( \mathbb{R} \) is replaced by an arbitrary field \( k \).

This identification of the stable isomorphism class \( w(\beta_0) \) with a local contribution to a degree answers a question of Eisenbud for polynomial maps of 1 variable. Eisenbud, in a survey article on his work with Levine, observed that \( w(\beta_0) \) is defined for a polynomial map \( f: \mathbb{A}^n_k \to \mathbb{A}^n_k \) over an arbitrary field \( k \) of odd characteristic, proposed \( w(\beta_0) \) as the definition of the degree, and asked if this degree has an interpretation, say in algebraic topology [Eis78, Some remaining questions (3)]. When \( n = 1 \), the Main Theorem identifies the class as a local contribution to the degree in \( \mathbb{A}^1 \)-homotopy theory.

In the companion paper [KW] the present authors extend this result to polynomial maps in several variables under the assumption \( k \) has odd characteristic:

**Theorem 2** (The main result of [KW]). The local \( \mathbb{A}^1 \)-degree of \( f: \mathbb{A}^n_k \to \mathbb{A}^n_k \) is the Eisenbud–Khimshiashvili–Levine class \( w(\beta_0) \).

The local \( \mathbb{A}^1 \)-degree is defined in terms of Morel’s degree homomorphism in the natural manner. Namely, the Purity theorem [MV99, Proposition 2.17, Theorem 2.23] identifies the quotient \( A^0_k \mathbb{A}^n_k / \mathbb{A}^n_k \) and more generally \( u_{\mathbb{U}} \) for \( U \subset \mathbb{A}^n_k \) open, with the motivic sphere \( (\mathbb{P}^1_k)^m \), so if \( f: \mathbb{A}^n_k \to \mathbb{A}^n_k \) is a polynomial map with an isolated zero at the origin, then we
can pick a Zariski neighborhood $U \subset A^n_k$ that excludes all zeros of $f$ other than the origin and define the local $A^1$-degree $\deg_{0}^{A^1}(f)$ to be the $A^1$-degree of the induced morphism

$$\left( \mathbb{P}^1_k \right)^{\wedge n} \cong \frac{U}{U - \{0\}} \to \frac{A^n_k}{A^n_k - \{0\}} \cong \left( \mathbb{P}^1_k \right)^{\wedge n}.$$ 

Finally, we wish to point out a subtle but significant point about the Main Theorem. While Scheja–Storch construct an explicit symmetric bilinear form $\beta$, Eisenbud–Levine emphasize working with the stable isomorphism class $w(\beta)$ of that form, and observe that there are other natural representatives of that class: For any $k$-linear function $\eta' : \mathbb{Q}(\mathbb{F}) \to k$ that coincides with the Scheja–Storch functional on the socle, the form $\beta'(b_1, b_2) := \eta'(b_1 \cdot b_2)$ is isomorphic to $\beta$ [EL77, Proposition 3.5]. However, when $n = 1$ interesting topological information is lost in passing from $\beta$ to $w(\beta)$. The isomorphism class $w(\beta)$ determines the $A^1$-degree, and the $A^1$-degree determines the pointed homotopy class for $n \geq 2$ but not for $n = 1$. For $n = 1$, the pointed homotopy class is determined by the $A^1$-degree together with a distinguished scalar representing the discriminant $\text{disc}(\beta)$. The scalar, and hence the homotopy class, can be recovered from $\beta$ because the scalar is the determinant of the Gram matrix of $\beta$ with respect to a natural basis, the basis dual to $1/g(x), x/g(x), x^2/g(x), \ldots, x^{\mu - 1}/g(x)$.

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References


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