Notes on Abelian Schemes

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These notes are a summary of some of the fundamental facts concerning abelian schemes. Most proofs have been omitted. Full proofs for most of the results discussed in these notes can be found in Néron Models by Bosch, Lütkebohmert, and Raynaud [1] and in chapter 6 of Geometric Invariant Theory by Mumford, Fogarty, and Kirwan [3]. Another good source for the material in these notes is Mumford’s Abelian Varieties [5].

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1 Abelian Schemes

Let $S$ be an arbitrary scheme. An abelian scheme over $S$ is an $S$-group scheme $A \to S$ that is proper, flat, finitely-presented, and has smooth and connected geometric fibers.

When $S = \text{Spec}(k)$ is the spectrum of a field, this is the standard definition of an abelian variety.

Remark. The function $s \to \dim A_s$ is locally constant in the Zariski topology. We typically assume that it is constant and equal to $g$.

Example. Some basic examples of abelian schemes are as follows:

1. If $C \to S$ is a proper, smooth $S$-curve with geometrically connected fibers, then $J = \text{Pic}^0_{C/S}$ is an abelian scheme called the relative Jacobian of $C/S$.

2. Suppose that $S$ is a connected Dedekind scheme (for example, the spectrum of a Dedekind ring or a regular curve over a field). Let $\eta$ be the generic point of $S$. Given an abelian variety $A_\eta$ over the generic point, it is a theorem of Néron that if $A_\eta$ extends to an abelian scheme $A_U$ over a non-empty open subset $U$ of $S$, then this extension is unique and functorial in $A_\eta$. It follows from uniqueness and “denominator-chasing” that $A_\eta$ extends over a maximal open subset.
The Weil Extension Lemma combined with the valuative criteria of properness implies that for any abelian scheme \( A \rightarrow S \) over a base \( S \) that is normal, connected, and locally noetherian and a smooth and any separated \( S \)-scheme \( Z \rightarrow S \), the natural map \( \text{Hom}_S(Z, A) \rightarrow \text{Hom}_\eta(Z_\eta, A_\eta) \) is bijective.

An isogeny \( f : A' \rightarrow A \) of abelian \( S \)-schemes is a surjective \( S \)-group map that is quasi-finite (has finite fibers). The Miracle Flatness Theorem (see homework 1) together with “proper + quasi-finite \( \Rightarrow \) finite” imply that \( f \) is finite and locally free.

**Theorem 1.** Any abelian scheme \( A/S \) is commutative. Any \( S \)-scheme map \( A \rightarrow G \) to a separated \( S \)-group scheme that maps the identity to the identity is a \( S \)-group homomorphism.

**Proof.** Begin by reducing to the case where \( S \) is the spectrum of an Artin local ring. To make this reduction, one uses properness, the Krull Intersection Theorem, and lim formalism from EGA IV3. Implicitly, we make use of the fact that the identity section of \( G \) is cut out by a quasi-coherent sheaf of ideals. This is where the hypothesis that \( G \) is separated is necessary.

The case where \( S \) is the spectrum of an algebraically closed field is classical (see Mumford [5], Chapter 2). The case where \( S \) is the spectrum of a possibly non-algebraically closed field immediately follows since extending scalars from \( k \) to \( \overline{k} \) defines a faithful functor.

This proves the result on the actual fibers (rather than just on the geometric fibers) over points \( s \in S \). By “taking differences”, we are reduced to proving:

**Lemma 1** (GIT Lemma). Suppose that \( S \) is the spectrum of an Artin local ring with closed point \( s \in S \) and that we are given a diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\pi} & \ & \downarrow{\eta} \\
S & \end{array}
\]

with \( \pi \) proper and flat. Let \( e \in X(S) \). Assume that \( H^0(X_s, \mathcal{O}_{X_s}) = k(s) \) (e.g. \( X_s \) is connected and geometrically reduced). If \( f(X_s) = \{ \text{point} \} \), then there exists \( \eta \in Y(S) \) such that \( f = \eta \circ \pi \).

**Proof.** By the theory of cohomology and Base Change (see chapter 3 of Hartshorne), it follows that \( \mathcal{O}_S \simeq \pi_*(\mathcal{O}_X) \). On the level of topology define the section \( \eta \) to be the continuous map \( f \circ e : |S| \rightarrow |Y| \). Check that defining \( \eta^# \) to be the composition

\[
\mathcal{O}_Y \xrightarrow{f^#} f_*\mathcal{O}_X = (\eta \circ \pi)_*\mathcal{O}_X = \eta_* \circ \pi_*\mathcal{O}_X \simeq \eta_*\mathcal{O}_X
\]

defines a morphism \( S \rightarrow Y \) satisfying the desired properties. \( \square \)
Corollary 1. For $N \geq 1$, the morphism $[N]_A : A \to A$ given by multiplication-by-$N$ is an isogeny of degree $N^{2g}$. Here $g =$ relative dimension of $A$. In particular, $A[N]$ is a finite, locally free $S$-group scheme of order $N^{2g}$. Furthermore, the system $A[\ell^n] = \{A[\ell^n]\}_n$ defines an $\ell$-divisible group of height $2g$.

Proof. This is Homework 3, exercise 5.

Exercise. Some basic facts about the $\ell$-divisible group of an abelian scheme:

1. Let $A/k$ be a $g$-dimensional abelian scheme over a field $k$. For $\ell \neq \text{char}(k)$, we have that $A[\ell]\ell$ is an étale $\ell$-divisible group of height $2g$. In particular, $A[\ell]\ell$ is the “same” as the Tate module $T_\ell(A) = \lim \leftarrow A[\ell^n](k_s)$ as a finite free $\mathbb{Z}_\ell$-module of rank $2g$ with continuous Galois action.

2. Let $k$ be a perfect field of characteristic $p > 0$. Then the $p$-divisible group $A[p^\infty]$ is the “same” as the Dieudonné module $D(A[p^\infty]) = \lim \leftarrow D(A[p^n])$. This is a module over the Dieudonné ring, $D_k$, that is finite and free of rank $2g$ over $W(k)$.

3. In characteristic $p$, the Serre-Tate Equivalence implies that the connected component of $A[p^\infty]$ is isomorphic to $\hat{O}_A$, $0$, the formal group of $A$. This is in homework set 3. Suppose that $f : A \to A'$ is an isogeny. Since $f$ is finite and locally free, it follows that $\ker(f)$ is a finite, locally free $S$-group scheme. The rank of $\ker f$ is locally constant over $S$. We call it the degree of $f$.

Consider the case where the degree of $f$ is constant and equal to $N$. By Deligne’s Theorem (see problem 1 on homework set 2) the kernel of $f$, $\ker(f)$, is killed by $[N^{2g}]_A$. One can use considerations with fppf sheaves to prove that there exists an isogeny $f' : A' \to A$ such that $f' \circ f = [N^{2g}]_A$ and $f \circ f' = [N^{2g}]_A'$. In particular, there exists an isogeny $f' : A' \to A$. We say that $A$ and $A'$ are isogenous if there exists an isogeny $f : A \to A'$. We have just shown that the property of being isogenous is an equivalence relation.

Remark. There is a wonderful Theorem of Raynaud that says that any finite locally free commutative group scheme $G \to S$ is Zariski locally on $S$ a closed subgroup of a relative Jacobian. This provides a foundation for crystalline Dieudonné theory for such group schemes by using Dieudonné theory for abelian schemes.

2 Abelian Varieties

Theorem 2. All abelian varieties over a field are projective (i.e. admit an ample line bundle).

Proof. Mumford [5] proves this in the case where the field is algebraically closed. A trick shows that if $X$ is a proper $k$-scheme and $X_\bar{k}$ is projective, then $X$ is projective.
Theorem 3. Let $A$ and $A'$ be abelian varieties over a field $k$. For any prime $\ell$, the natural map

$$t_\ell : \mathbb{Z}_\ell \otimes \mathbb{Z} \text{Hom}_k(A, A') \to \text{Hom}_k(A[\ell\infty], A'[\ell\infty])$$

induced by functoriality is injective.

Proof. First reduce to the case where $k$ is algebraically closed. When $\ell \neq \text{char}(k)$, the proof can be found in Mumford’s book [5]. For $\ell = \text{char}(k)$, the same argument goes through with the Dieudonné module on the geometric fiber playing the role of the Tate module.

Corollary 2. If $f \in \text{End}_k(A)$, set $P_f(n) = \deg ([n]_A - f)$. This is understood to be 0 if $[n]_A - f$ is not an isogeny. Then $P_f$ is a polynomial in $n$ with integer coefficients. It is a monic of degree $2g$. Let $k'/k$ be any perfect extension. The polynomial $P_f$ is equal to characteristic polynomial of the induced endomorphism on

$$\begin{cases} 
T_\ell(A) & \ell \neq \text{char}(k) \\
\mathbb{D}(A_k[p\infty]) & \ell = \text{char}(k)
\end{cases}$$

Here we consider $T_\ell(A)$ as a $\mathbb{Z}_\ell$-module and $\mathbb{D}(A_k[p\infty])$ as a $W(k')$-module.

Proof. First reduce to the case where $k$ is algebraically closed and $k = k'$. For $\ell \neq p$, a proof can be found on pages 180-181 of Mumford’s Abelian Varieties [5]. For $\ell = p$, the same proof carries over by using Dieudonné modules.

Using theorem 3 and some cleverness, one can show that $\text{Hom}_k(A_k, A'_k)$ is finitely generated over $\mathbb{Z}$ and of rank at most $4 \dim A \dim A'$. It follows that the same holds for $\text{Hom}_k(A, A')$.

Corollary 3 (Riemann Hypothesis). Suppose $k$ is finite and of cardinality $q$. If $f = F_{A/k}$ is the relative Frobenius morphism, then all the complex roots of $P_f$ have absolute value $\sqrt{q}$ (i.e. are “Weil $q$-numbers”).

Proof. See pages 203-207 of Mumford [5].

Theorem 4 (Tate’s Conjecture). If $k$ is finitely generated over the prime field, then the map $t_\ell$ is an isomorphism.

Proof. This was proven by Tate in the case where $k$ is a finite field. Zahrin extended this result to the case where $k$ is finitely generated over a finite field. Building on Tate’s method, Faltings proved the result in the case where $k$ is a number field and later extended his proof to the general case where $k$ is finitely generated over $\mathbb{Q}$.
3  Duality Theory

We will now discuss the duality theory of abelian schemes. First, we define Picard schemes. A Picard scheme is a certain space that parameterizes line bundle on a fixed scheme and is defined for a fairly general class of schemes. For the remainder of this section, we will let $\pi : X \to S$ be a proper, flat, finitely-presented morphism with $\pi_* (\mathcal{O}_X) = \mathcal{O}_S$ holding “universally”. Furthermore, let $e : S \to X$ be a section of $\pi$. We will be particularly interested in the case where $X/S$ is an abelian variety and $e$ is the identity section.

Given an $S$-scheme $T$, we define a functor $\text{Pic}_{X/S,e}(\_)$ by:

$$\text{Pic}_{X/S,e}(T) = \{(\mathcal{L}, i) : \mathcal{L} \text{ is an invertible sheaf on } X_T, i = e^*_T(\mathcal{L}_T) \simeq \mathcal{O}_T\}/\cong$$

Here $\cong$ indicates that isomorphic pairs $(\mathcal{L}, i)$ are identified. However, the hypotheses on $\pi$ are set up so that an object $(\mathcal{L}, i)$ has no non-trivial automorphisms. This ensures that no real information is lost in passing from the category of rigidified line bundles over $X_T$ to the set of isomorphism classes. In particular, the fact that a rigidified line bundle $(\mathcal{L}, i)$ has no non-trivial automorphisms implies that assignment $T \mapsto \text{Pic}_{X/S,e}(T)$ defines a Zariski sheaf of abelian groups, called the relative Picard functor.

Exercise. The absolute Picard group of a scheme $Y$, denoted $\text{Pic}(Y)$, is defined to be the group of isomorphism classes of line bundles on $Y$. Given a line bundle $\mathcal{L}$ over $X_T$, define $\tau_e(\mathcal{L}) = \mathcal{L} \otimes \pi^*_T(e^*_T(\mathcal{L}^{-1})$. Here $\pi$ denotes projection onto $e$. Given the existence of this trivialization, the assignment $\mathcal{L} \mapsto (\tau_e(\mathcal{L}), i_{\text{can}})$ defines a functorial homomorphism $\text{Pic}(X_T) \to \text{Pic}_{X/S,e}(T)$

Prove that the kernel of this homomorphism is $\pi^*_T(\text{Pic}(T))$.

There are several general theorems that assert the existence of the Picard scheme of a scheme under suitable hypotheses. For our purposes, the following theorem is more than sufficient.

Theorem 5 (Grothendieck-Oort). If $S = \text{Spec}(k)$ and $X \to S$ satisfies the hypothesis stated at the beginning of this section, then $\text{Pic}_{X/k,e}$ is representable by a locally finite type $k$-group $\text{Pic}_{X/k,e}$. This scheme is a disjoint union of quasi-projective $k$-schemes.

We let $\varphi$ denote the universal line bundle on $X \times \text{Pic}_{X/k,e}$. This bundle comes equipped with a canonical trivialization $(e \times 1)^*(\varphi) \simeq \mathcal{O}_{X/k,e}$. The connected component of $\text{Pic}_{X/k,e}$ containing the identity is denoted $\text{Pic}_{X/k,e}^0$. The restriction of $\varphi$ to the scheme $\text{Pic}_{X/k,e}^0$ is denoted $\varphi^0$ and is called the Poincaré bundle. In general, $\text{Pic}_{X/k,e}^0$ is geometrically connected and quasi-compact.

Exercise. 1. If $X_k$ is smooth, then $\text{Pic}_{X/k,e}^0$ is proper (and hence projective over $k$). Hint: use the valuative criterion.
2. The scheme $\text{Pic}_X/k,e$ can be non-smooth even when $X/k$ is smooth. In fact, one can take $X$ to be a surface over a field of characteristic $p$. Hint: This is highly non-trivial. Examples can be found among the surfaces discussed in Igusa’s paper [2]. For more details, see Mumford’s book Lectures on Curves on an Algebraic Surface [4]

We now specialize to the case of an abelian variety $A/k$.

**Theorem 6.** If $A/k$ is an abelian variety, then the Picard scheme $A^\vee/k = \text{Pic}^0_{A/k,e}/k$ is smooth and hence an abelian variety.

We call $A^\vee$ the **dual abelian variety** of $A$.

**Remark:** By construction, the Poincaré sheaf $\mathcal{P}^0$ has a canonical trivialization over $\{(e) \times A^\vee\}$. The Poincaré sheaf also has a distinguished trivialization over $A \times \{e^\vee\}$. The argument for this is as follows. By the functorial definition of $\text{Pic}^0_{X/T,e}$, the restriction of $\mathcal{P}^0$ to $A \times \{e^\vee\}$ is equal to the identity element of $\text{Pic}^0_{A/k,e}(k)$. Now the identity element of this group is the trivial bundle on $A$ equipped with its canonical trivialization. In particular, the restriction of $\mathcal{P}^0|_{A \times \{e^\vee\}}$ has a distinguished trivialization.

The bundle $\mathcal{P}^0$ equipped with this distinguished trivialization over $A \times \{e^\vee\}$ induces a morphism $\kappa_A: A \rightarrow (A^\vee)^\vee$. The morphism $\kappa_A$ of abelian varieties maps the identity section to the identity section. By theorem 1, any such morphism is a homomorphism of abelian varieties.

**Theorem 7.** The morphism $\kappa_A$ is an isomorphism

**Proof.** Reduce to the case where $k$ is algebraically closed. This case is covered by Mumford [5].

Given $\phi: A \rightarrow A^\vee$, there is an induced morphism $(A^\vee)^\vee \rightarrow A^\vee$. The homomorphism $\kappa_A$ can be used to identify $A$ with $(A^\vee)^\vee$. Once this is done, we obtain a morphism $A \rightarrow A^\vee$ called the **dual map**, denoted $\phi^\vee$. We say that $\phi$ is **symmetric** if $\phi = \phi^\vee$.

A **polarization** of $A$ is a symmetric isogeny $\phi: A \rightarrow A^\vee$ with the property that $(1, \phi)^*(\mathcal{P})$ is ample. A polarization is said to be a **principal polarization** if it is of degree 1.

A symmetric homomorphism of an abelian variety is analogous to a symmetric bilinear form on a finite dimensional real or rational vector space. Under this analogy, a polarization is analogous to a positive-definite quadratic form on such a vector space. The Poincaré line bundle corresponds to the evaluation pairing $V \times V^\vee \rightarrow k$ on a vector space. In the complex-analytic theory of abelian varieties, this analogy can be made more precise by relating polarizations to bilinear pairings on $H_1(A(\mathbb{C}), \mathbb{Z})$.

In the general algebraic setting, the homology group $H_1(A(\mathbb{C}), \mathbb{Z})$ is replaced with the Tate module. Fix a prime $\ell$ not equal to the characteristic. By definition, there is an evaluation pairing $A[\ell^\infty] \times (A[\ell^\infty])^\vee \rightarrow \mu_{\ell^\infty}$. Since $\ell$ is not equal to the characteristic of the ground field, this pairing can be identified with the evaluation pairing $e: T_\ell(A) \times T_\ell(A)^\vee \rightarrow \mathbb{Z}_\ell(1)$. 

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Given a symmetric homomorphism $\phi : A \to A^\vee$, there is an induced map on the $\ell$-divisible groups $A[\ell^\infty] \to A^\vee[\ell^\infty]$. One can show that $A^\vee[\ell^\infty]$ can be identified with the Cartier dual $(A[\ell^\infty])^\vee$ of $A[\ell^\infty]$. By universal formalism, giving a homomorphism from an $\ell$-divisible group to its Cartier dual is equivalent to giving a bilinear pairing $e_\phi : A[\ell^\infty] \times A[\ell^\infty] \to \mu_{\ell^\infty}$. Since $\ell$ is not equal to the characteristic, the $\ell$-divisible groups $A[\ell^\infty]$ and $\mu_{\ell^\infty}$ can be identified with their groups of geometric points with the induced Galois action. After making this identification, we obtain a bilinear pairing $e_\phi : T_\ell(A) \times T_\ell(A) \to \mathbb{Z}/\ell^2(1)$ called the **Weil pairing**. This pairing is the pairing induced by $e$ via $\phi$ in the sense that $e_\phi(x, y) = e(x, \phi(y))$. When $\phi$ is a polarization, it can be shown that $e_\phi$ is non-degenerate. However, one should beware that the pairing $e_\phi$ is skew-symmetric, not symmetric.

Using the classification of simple, finite, connected, commutative $p$-divisible groups over an algebraically closed field, one can describe all polarizations $\phi$ of an abelian variety over $\overline{k}$ and deduce that $\deg \phi$ is a perfect square. In the complex-analytic case, this property follows from the fact that a non-degenerate symplectic space over the integers has square determinant (via Pfaffians).

The notion of the dual abelian scheme can be defined over a fairly general base $S$. In this generality, one can still define what is meant by a polarization. The fiber-wise degree of a polarization of an abelian scheme is (Zariski-)locally constant on the base $S$. Given this formalism, one can prove the following theorem:

**Theorem 8** (Mumford). Fix integers $g, d, N \geq 1$. For any $\mathbb{Z}[1/N]$-scheme $S$, let $M_{g,d,N}(S)$ be the set of isomorphism classes of triples $(A, \phi, i)$, where $A$ is an abelian scheme over $S$ of relative dimension $g$, $\phi$ is a degree $d^2$ polarization on $A$, and $i : (\mathbb{Z}/NZ)^{2g} \to A[N]$ is an isomorphism of $S$-groups. Such triples admit no non-trivial automorphisms for $N \geq 3$, and for such $N$ the functor $M_{g,d,N}$ is represented by a quasi-projective $\mathbb{Z}[1/N]$-scheme. In particular, up to isomorphism, over any finite field there are only finitely many $g$-dimensional abelian varieties equipped with a polarization of degree $d^2$.

**References**


