THE KINEMATIC FORMULA IN RIEMANNIAN HOMOGENEOUS SPACES

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Abstract

Let G be a Lie group and K a compact subgroup of G. Then the homogeneous space G/K has an invariant Riemannian metric and an invariant volume form Ω_G. Let M and N be compact submanifolds of G/K, and $I(M \cap gN)$ an "integral invariant" of the intersection $M \cap gN$. Then the integral

$$\int_G I(M \cap gN)\, \Omega_G(g) \tag{1}$$

is evaluated for a large class of integral invariants I. To give an informal definition of the integral invariants I considered, let $X \subset G/K$ be a submanifold, h^X the vector valued second fundamental form of X in G/K. Let $\mathcal{P}$ be an "invariant polynomial" in the components of the second fundamental form of h^X. Then the integral invariants considered are of the form

$$I^{\mathcal{P}}(X) = \int_X \mathcal{P}(h^X)\, \Omega_X.$$

If $\mathcal{P} \equiv 1$ then $I^{\mathcal{P}}(M \cap gN) = \mathrm{Vol}(M \cap gN)$. In this case the integral (1) is evaluated for all G, K, M and N.

For $\mathcal{P}$ of higher degree the integral (1) is evaluated when G is unimodular and G is transitive on the set on tangent spaces of each of M and N. Then, given $\mathcal{P}$, there is a finite set of invariant polynomials $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ (depending only on $\mathcal{P}$) so that for all appropriate M and N

$$\int_G I^{\mathcal{P}}(M \cap gN)\, \Omega_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N). \tag{2}$$

This generalizes the Chern-Federer kinematic formula to arbitrary homogeneous spaces with an invariant Riemannian metric and leads to new formulas even in the case of submanifolds of Euclidean space.

The approach used here also leads to a "transfer principle" that allows integral geometric formulas to be moved between homogeneous spaces that have the same isotropy subgroups. Thus if G/K and G'/K' are homogeneous spaces with both G and G' unimodular and the subgroups K and K' are isotropic equivalent, then any integral geometric formula of the form (2) that holds for submanifolds of G/K also holds for submanifolds of G'/K'. In particular the transfer principle shows that the Chern-Federer holds in all simply connected space forms of constant sectional curvature and not just in Euclidean space.

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1. Introduction

Let G be a Lie group and K a closed subgroup of G. If M and N are compact submanifolds of the homogeneous space G/K. Then a good deal of energy in integral geometry has gone into computing integrals of the following type

$$\int_G I(M \cap gN)\, \Omega_G(g) \tag{1-1}$$

where I is an "integral invariant" of the submanifold $M \cap gN$. For example in the case that G is the group of isometries of Euclidean space $\mathbb{R}^n$, M and N are submanifolds of $\mathbb{R}^n$ and $I(M \cap gN) = \mathrm{Vol}(M \cap gN)$ then evaluation of (1-1) leads to formulas due to Poincaré, Blaschke and others (see the book [18] for references) or in the same case if we let $I(M \cap gN)$ be one of the integral invariants arising from the Weyl tube formula then the evaluation of (1-1) gives the kinematic formula of Federer [8] and Chern [6]. In the case G is the unitary group $U(n+1)$ acting on complex projective space $\mathbb{CP}^n$ and M and N are complex analytic submanifolds of $\mathbb{CP}^n$ then letting $I(M \cap gN) = \mathrm{Vol}(M \cap gN)$ in (1-1) leads to results of Santaló [17] or letting $I(M \cap gN)$ be the integral of a Chern class leads to the recent kinematic formula of Shifrin [19]. In this paper we will assume that G/K has an invariant Riemannian metric and evaluate (1-1) for arbitrary M and N in the case that $I(M \cap gN) = \mathrm{Vol}(M \cap gN)$ (this generalizes the results of Brothers [2]) and for "arbitrary" integral invariants I in the case G is unimodular and acts transitively on the sets of tangent spaces to each of M and N. That is we will give a definition of integral invariant general enough to cover most cases that have come up to date and for $I(M \cap gN)$ one of these invariants we will evaluate (1-1) in terms of the integral invariants of M and N. This leads to new formulas (at least modulo evaluating some constants) even for submanifolds of Euclidean space $\mathbb{R}^n$.

Before giving a summary of our results we give a reasonably exact statement of our results for submanifolds of Euclidean space. This should make what follows more concrete. Recall that if M^p is a p dimensional submanifold of $\mathbb{R}^n$ and $x \in M$ then the second fundamental form h^M_x of M at x is a symmetric bilinear map from $TM_x \times TM_x$ to $T^\perp M_x$ (here TM is the tangent bundle of M and $T^\perp M$ is the normal bundle of M in $\mathbb{R}^n$). If $e_1, \dots, e_n$ is an orthonormal basis of $\mathbb{R}^n$ such that $e_1, \dots, e_p$ is a basis of TM_x and $e_{p+1}, \dots, e_n$ is a basis of $T^\perp M_x$ then the components of h^M_x in this basis are the numbers $(h^M_x)^\alpha_{ij} = \langle h^M_x(e_i, e_j), e_\alpha\rangle$ $1 \le i, j \le p$, $p+1 \le \alpha \le n$ where $\langle\, ,\rangle$ is the usual inner product on $\mathbb{R}^n$.

Call a polynomial $\mathcal{P}(X^\alpha_{ij})$ in variables X^α_{ij} $1 \le i, j \le p$, $p+1 \le \alpha \le n$ and $X^\alpha_{ij} = X^\alpha_{ji}$ which is invariant under the substitutions

$$X^\alpha_{ij} \mapsto \sum_{s,t,\beta} a_{is} a_{jt} X^\beta_{st} b_{\alpha\beta} \tag{1-2}$$

for all p by p orthogonal matrices $[a_{ij}]$ and all $(n-p)$ by $(n-p)$ orthogonal matrices $[b_{\alpha\beta}]$ **an invariant polynomial defined on the second fundamental forms of p dimensional submanifolds**. If $\mathcal{P}$ is such a polynomial then

$$\mathcal{P}(h^M_x) = \mathcal{P}((h^M_x)^\alpha_{ij})$$

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is defined independently of the choice of the orthonormal basis $e_1, \dots, e_n$. For each such polynomial define an integral invariant $I^{\mathcal{P}}$ on compact p dimensional submanifolds of $\mathbb{R}^n$ by

$$I^{\mathcal{P}}(M) = \int_M \mathcal{P}(h_x^M)\,\Omega_M(x)$$

where Ω_M is the volume density on M. Using the invariance of $\mathcal{P}$ under the substitution (1-2) it follows that $I^{\mathcal{P}}$ has the basic invariance property $I^{\mathcal{P}}(gM) = I^{\mathcal{P}}(M)$ for all isometries g of $\mathbb{R}^n$. This set of invariants contains a large number of the integral invariants which occur in geometry.

We now state the kinematic formula:

THEOREM. *Let p, q be integers with $1 \leq p, q \leq n$ and $p+q \geq n$. Let $\mathcal{P}$ be an invariant polynomial defined on the second fundamental forms of $p+q-n$ dimensional submanifolds and assume that $\mathcal{P}$ is homogeneous of degree $\leq p+q-n+1$. Then there is a finite set of pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ such that:*

1. *Each $\mathcal{Q}_\alpha$ is a homogeneous invariant polynomial on the second fundamental forms of p dimensional submanifolds,*
2. *Each $\mathcal{R}_\alpha$ is a homogeneous invariant polynomial on the second fundamental forms of q dimensional submanifolds,*
3. *For each α $\operatorname{degree} \mathcal{Q}_\alpha + \operatorname{degree} \mathcal{R}_\alpha = \operatorname{degree} \mathcal{P}$,*
4. *For all compact p dimensional submanifolds M and q dimensional submanifolds N of $\mathbb{R}^n$ (each possibly with boundary)*

$$\int_G I^{\mathcal{P}}(M \cap gN)\,\Omega_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N) \tag{1-3}$$

where G is the group of isometries of $\mathbb{R}^n$ and Ω_G its invariant measure.

Once the group theoretic ideas involved in proving this have been isolated it becomes no harder to prove (1-3) for submanifolds M and N of an arbitrary Riemannian homogeneous space G/K provided only that G is unimodular and G is transitive on the sets of tangent spaces to each of M and N. One advantage to working in this generality is that it becomes clear that the form of kinematic formulas in a homogeneous space G/K does not depend on the full group of motions G, but only on the invariant theory of the isotropy subgroup K. This this observation leads to a "transfer principle" allows us to "move" kinematic formulas proven for a homogeneous space G/K to any other homogeneous space with an isotropy subgroup equivalent to K. For example, the Chern-Federer kinematic formula for submanifolds of $\mathbb{R}^n$ is

$$\int_G \mu_{2l}(M^p \cap gN^q)\,\Omega_G(g) = \sum_{k=0}^{l} c(n,p,q,l,k)\mu_{2k}(M^p)\mu_{2(l-k)}(N^q) \tag{1-4}$$

where the μ's are the integral invariants from the Weyl tube formula (defined in section 10 below), G is the group of isometries of $\mathbb{R}^n$ and $c(n,p,q,l,k)$ is a constant only depending on the indicated parameters. The transfer principle tells us this formula holds in all simply connected spaces of constant sectional curvature (the sphere and the hyperbolic space form) with the same values for the constants $c(n, p, q, l, k)$. (Here the integrand in the definition of $\mu_{2k}(M)$ must be expressed—and this is an important point—as a polynomial in the components of the second fundamental form of M and not as a polynomial in the components of the curvature tensor of M.)

We now summarize our results. In section 2 we prove our "basic integral formula" for submanifolds M and N of a Lie group G on which all our latter integral formulas will be based. The idea behind its proof is extremely simple: Apply the Federer coarea formula to the function $f : M \times N \to G$, given by $f(\xi, \eta) = \xi\eta^{-1}$, and interpret the result geometrically. Although the details are quite different the proofs are very much in the style of the papers of Federer [8] and Brothers [2] to which the present paper is greatly indebted. One big difference between the proofs here and those in [2] and [8] is that we work in the smooth category and thus avoid the measure theoretic problems which Federer and Brothers have to deal with. In section 3 the integral (1-1) is evaluated for any compact submanifolds M and N of a Riemannian homogeneous space G/K in the case $I(M \cap gN) = \mathrm{Vol}(M \cap gN)$ and examples are given of how the transfer principle in this context can be used to compute the various constants occurring in the formulas in an efficient and elegant manner. The proofs here precede by applying the integral formula of section 2 to the submanifolds $\pi^{-1}M$ and $\pi^{-1}N$ of G (where $\pi : G \to G/K$ is the natural projection) and then "pushing" the result of this back down to G/K. In an appendix to this section a general Crofton type formula is proven. Apart from its own interest this Crofton formula lets us to identify the invariant measures used in Chern's paper [4] with Riemannian invariants. This allows examples to be given of homogeneous spaces (in particular $\mathbb{CP}^2$) where these measures are different from the Riemannian volume of the submanifold and where these measures are not unique, so that the choice of the measure to be used is determined by the type of integral geometric formula to be proven.

In section 4 we give a general definition of an integral invariant of a compact submanifold M of a Riemannian homogeneous space G/K. Once this has been done the general kinematic formula and the transfer principle are stated and in an appendix a general analogue of the "linear" kinematic formulas in section 8 of [6] and section 3 of [19] is given. The next two sections contain the lemmas needed to prove these results. In particular, section 5 gives the needed results on the geometry of intersections of submanifolds and section 6 gives the required algebraic facts and definitions.

In section 7 a restatement of the kinematic formula is given in terms of the algebraic definitions of section 6. This new form of the kinematic formula makes the results of section 4 quite transparent and is better adapted to concrete calculations. This theorem is then proven. As with the results of section 3 the proof proceeds by replacing the submanifolds M and N of G/K by $\pi^{-1}M$ and $\pi^{-1}N$ ($\pi : G \to G/K$ natural projection), using the basic integral formula of section 2, and then pushing the result back down to G/K.

The next section gives a proof that for spaces of constant sectional curvature the integrals involved in equation (1-3) converge when $\mathrm{degree}(\mathcal{P}) \leq p + q - n + 1$. The main tool in the proof is a formula from Chern's paper [6].

The last two sections of the paper are devoted to giving a new proof of the

Chern-Federer kinematic formula (1-4) which works in all simply connected spaces of constant sectional curvature. This could be done by using the transfer principle to "move" the result from $\mathbb{R}^n$, where it is known, to the other space forms. However, this does not lead to any new insights. The idea in our proof of (1-4) is to give an algebraic characterization of the polynomials appearing as the integrands of the μ's, which is of interest in its own right, and which exhibits both the Weyl tube formula and (1-4) as consequences of the invariant theory of the orthogonal group.

In an appendix we give a short proof of the coarea formula for smooth maps which avoids the measure theoretic complications arising in the case of Lipschitz maps.

It is worth remarking at this point that the methods used here seem to be best adapted to proving integral geometric formulas involving purely Riemannian invariants. For example it is possible to give a proof of the main result of Shifrin's paper [19] in the style of the proof given here of the Chern-Federer kinematic formula. This can be done (at least for complex hypersurfaces) by giving a characterization of the integral invariants arising in the formula for the volume of a tube about a complex analytic submanifold of $\mathbb{CP}^n$ similar to the one given in section 9 below for the μ's (see [23], [10] or [12] for the tube formula in $\mathbb{CP}^n$). The resulting formula is in terms of integrals over the submanifolds of invariant polynomials in the components of the second fundamental forms. But then one of the prettiest facts about these invariants becomes almost invisible, they are also integrals of Chern forms which represent cohomology classes on the submanifolds. The proof in [19] not only makes this clear, it uses this fact strongly in the proof. On the other hand, there are Riemannian integral invariants $I^{\mathcal{P}}$ of complex analytic submanifolds of $\mathbb{CP}^n$ which are not covered by the theorems in [19] (he only considers invariant polynomials in the Chern forms and the Kaehler form) for which (1-1) can be evaluated by the methods given here.

Our notation and terminology is standard. By "smooth" we mean of class C^∞. If M is a smooth manifold then TM is its tangent bundle and TM_x its tangent space at x. If $f : M \to N$ is a smooth map between manifolds then $f_{*x} : TM_x \to TN_{f(x)}$ is the derivative of f at $x \in M$. If M and N are Riemannian manifolds then $f : M \to N$ is a Riemannian submersion iff for all $x \in M$ the derivative $f_{*x} : TM_x \to TN_{f(x)}$ is surjective and f_{*x} restricted to the orthogonal complement of kernelf_{*x} is a linear isometry. In the case $\dim(M) = \dim(N)$ then a Riemannian submersion is a local isometry. We regard discrete subsets S of a manifold as submanifolds of dimension zero in which case the volume of S is defined to be the number of points in S. Lastly, if $f : M \to N$ is an immersed submanifold of M, then we will repress the immersion f and just say that "N is a submanifold of M". In this case the tangent spaces to N will be identified with subspaces to tangent spaces to M in the natural way.

It is my pleasure to acknowledge, first of all, Ted Shifrin who spent a morning explaining the results of [19] to me. This got me hooked on the idea of trying to understand kinematic formulas in the context of Riemannian homogeneous spaces. This paper is very much a result of that conversation. I also would like to thank Paul Hewitt for some conversations on invariant theory which greatly speeded up my coming up with the correct formulation and proof of theorem 9.9. Finally the referee made a very thorough reading of the manuscript and made suggestions

leading to numerous improvements.

2. The Basic Integral Formula for Submanifolds of a Lie Group.

2.1 We start with a discussion on angles between subspaces. If V is an n dimensional and W is an m dimensional subspace of an inner product space with inner product $\langle\,,\rangle$ then let $v_1, \dots, v_n$ be an orthonormal basis of V and $w_1, \dots, w_m$ an orthonormal basis of W and define

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_n \wedge w_1 \wedge \cdots \wedge w_m\| \tag{2-1}$$

where

$$\|x_1 \wedge \cdots \wedge x_k\|^2 = \det(\langle x_i, x_j\rangle). \tag{2-2}$$

If V and W are both one dimensional then $\sigma(V, W) = |\sin\theta|$ where θ is the angle between V and W. In general $0 \leq \sigma(V, W) \leq 1$ with $\sigma(V, W) = 0$ if and only if $V \cap W \neq \{0\}$ and $\sigma(V, W) = 1$ if and only if V is orthogonal to W. Also if ρ is a linear isometry of the inner product space containing V and W into some other inner product space then

$$\begin{aligned} \sigma(\rho V, \rho W) &= \sigma(V, W) \\ \sigma(V, W) &= \sigma(V, W) \end{aligned} \tag{2-3}$$

2.2 Let G be a Lie group and $\xi \in G$. Then left and right translation by ξ on G will be denoted by L_ξ and R_ξ respectively, that is $L_\xi(g) = \xi g$ and $R_\xi(g) = g\xi$. Left translation can be used to identify all tangent spaces to G with TG_e, the tangent space to G at the identity element e. Assume that G has a left invariant metric $\langle\,,\rangle$ then this identification of the tangent spaces of G with each other allows the above definition of angles to be extended to compare angles between subspaces of tangent spaces to G at different points. To be exact if V is a subspace of TG_ξ and W is a subspace of TG_η then set

$$\sigma(V, W) = \sigma(L_{\xi^{-1}*}V, L_{\eta^{-1}*}W). \tag{2-4}$$

With this definition it follows that for all $g \in G$

$$\sigma(L_{g*}V, W) = \sigma(V, L_{g*}W) = \sigma(V, W). \tag{2-5}$$

Also if $a \in G$ and the metric is invariant by R_a then

$$\sigma(R_{a*}V, R_{a*}W) \qquad \text{if} \qquad R_a^*\langle\,,\rangle = \langle\,,\rangle \tag{2-6}$$

This follows from (2-3) with $\rho = R_{a*}$. By convention $\sigma(V, W) = 1$ if $V = \{0\}$ or $W = \{0\}$.

2.3 We now define the **modular function** Δ of G. Let $E_1, \dots, E_n$ (n = dimension of G) be any basis for the left invariant vector fields on G. Then, for each

$g \in G$, $R_{g^{-1}*}E_1, \ldots, R_{g^{-1}*}E_n$ is also a basis for the left invariant vector fields and thus

$$\|R_{g^{-1}*}E_1 \wedge \cdots \wedge R_{g^{-1}*}E_n\| = \Delta(g)\|E_1 \wedge \cdots \wedge E_n\| \tag{2-7}$$

for some positive real number $\Delta(g)$. From this definition it follows that Δ is a smooth homomorphism of G into the multiplicative group of positive real numbers. The following equivalent definition will also be used in the sequel. If ξ is any point of G and $u_1, \ldots, u_n$ any basis of TG_ξ then

$$\Delta(g)\|u_1 \wedge \cdots \wedge u_n\| = \|R_{g^{-1}*}u_1 \wedge \cdots \wedge R_{g^{-1}*}u_n\| \tag{2-8}$$

This follows from (2-7) by extending each u_i to a left invariant vector field on G.

2.4 Remark. A Lie group G is called unimodular if $\Delta \equiv 1$. It is well known that all compact groups, all semisimple groups and all nilpotent groups are unimodular.

2.5 Recall that if M and N are immersed submanifolds of some manifold S then M and N **intersect transversely** if and only if $x \in M \cap N$ implies $TM_x + TN_x = TS_x$ (here $TM_x + TN_x$ is the subspace of TS_x generated by TM_x and TN_x). If S has a Riemannian metric this is the same as requiring $T^\perp M_x \cap T^\perp N_x = \{0\}$. If M and N have nonempty intersection and intersect transversely then $M \cap N$ is a smooth submanifold of S whose dimension is $\dim M + \dim N - \dim S$.

2.6 A remark on notation. For any Riemannian manifold M we will denote the volume density on M by Ω_M. Then Ω_M can be thought of either as a measure on M or as the absolute value of one of the two locally defined volume forms on M. (See [25] page 53 for a more detailed discussion of densities.) In particular Ω_M and integration with respect to Ω_M are defined without any assumption about the orientablity of M. Despite this it will often be useful when doing calculations to assume that M is oriented and that Ω_M is one of the two volume forms on M. In all cases where it is convenient to do this the calculation is local, and thus we can restrict down to an oriented subset of M, do the calculation just as if Ω_M was a form and then take absolute values when we are done. This will be done without mention in the sequel and hopefully no confusion will result.

2.7 Basic integral formula. *Let G be a Lie group with a left invariant metric $\langle\,,\rangle$. Let M and N be immersed submanifolds (possibly with boundary) of G with* $\dim(M) + \dim(N) \geq \dim(G)$. *Then for almost all $g \in G$ the submanifolds M and gN intersect transversely and if h is any Borel measurable function on $M \times N$ such that the function $(\xi, \eta) \mapsto h(\xi, \eta)\Delta(\eta)$ is integrable on $M \times N$, then*

(2-9)

$$\int_G \int_{M\cap gN} h \circ \varphi_g \, \Omega_{M\cap gN}\, \Omega_G(g) = \iint_{M\times N} h(\xi,\eta)\Delta(\eta)\sigma(T^\perp M_\xi, T^\perp N_\eta)\, \Omega_{M\times N}(\xi,\eta)$$

where $\varphi_g : M \cap N \to M \times N$ is given by

$$\varphi_g(x) = (\xi, g^{-1}\xi). \tag{2-10}$$

2.8 Remarks. (1) The formula (2-9) is closely related to the formula of theorem 5.5 in the paper [2] of Brothers.

(2) It is possible that M and gN do not have nonempty transverse intersection for any $g \in G$ (in which case the set of $g \in G$ with $M \cap gN \neq \emptyset$ has measure zero and so (2-9) reduces to $0 = 0$). As an example of this let G be the additive group $\mathbb{R}^2$ and let M and N be segments parallel to the x-axis, say $M = \{(x, y_0) : a \leq x \leq b\}$, $N = \{(x, y_1) : c \leq x \leq d\}$. In this case it is still possible to give a version of (2-9) which gives a nonzero result. This is done by using the generalized coarea formula given in section 10 of the paper of Brothers just quoted in the proof of (2-9) at the places where we use the coarea formula. For details of this type of construction see section 11 of Brothers' paper and remark 3.10(2) below.

2.9 In proving the basic integral formula it can be assumed that M and N are embedded submanifolds of G. To see this use a partition of unity on $M \times N$ to restrict the support of h down to a subset of $M \times N$ of the form $U \times V$ where U is an open orientable submanifold of M with smooth boundary, V is an open orientable submanifold of N both U and V are embedded in G. Then prove (2-9) with M replaced by U and N replaced by V and then sum over the partition of unity.

For the rest of this section we will use the following notation $f : M \times N \to G$ is the function

$$f(\xi, \eta) = \xi\eta^{-1} \tag{2-11}$$

Then for all $g \in G$

$$f^{-1}[g] = \{(\xi, \eta) \in M \times N : f(\xi, \eta) = \xi\eta^{-1} = g\}.$$

By the coarea formula (see the appendix for the statement of this formula and for the definition of the Jacobian $Jf(\xi, \eta)$),

$$\int_G \int_{f^{-1}[g]} h\, \Omega_{f^{-1}[g]}\, \Omega_G(g) = \iint_{M\times N} h(\xi, \eta) Jf(\xi, \eta)\, \Omega_{M\times N}(\xi, \eta). \tag{2-12}$$

What we will do is compute the Jacobian $Jf(\xi, \eta)$ in terms of the geometric data (which will relate its value to the angle $\sigma(T^{\perp}M_\xi, T^{\perp}N_\eta)$) and show that for almost all $g \in G$ the map φ_g is a diffeomorphism of $M \cap gN$ with $f^{-1}[g]$ and use this to relate the integrals $\int_{f^{-1}[g]} h\, \Omega_{f^{-1}[g]}$ to the integrals $\int_{M\cap gN} h \circ \varphi_g\, \Omega_{M\cap gN}$.

In what follows we will use the standard isomorphism of $T(M \times N)_{(\xi,\eta)}$ with $TM_\xi \oplus TN_\eta$. Vectors in $T(M \times N)_{(\xi,\eta)}$ will be written as (X, Y) with $X \in TM_\xi$ and $Y \in TM_\eta$.

2.10 Lemma. *If* $(X, Y) \in T(M \times N)_{(\xi,\eta)}$ *then*

$$\begin{aligned} f_{*(\xi,\eta)}(X, Y) &= R_{\eta^{-1}*}X - R_{\eta^{-1}*}L_{\xi\eta^{-1}*}Y \\ &= R_{\eta^{-1}*}(X - L_{\xi\eta^{-1}*}Y) \end{aligned} \tag{2-13}$$

Proof. It is enough to show $f_{*(\xi,\eta)}(X, 0) = R_{\eta^{-1}*}X$ and $f_{*(\xi,\eta)}(0, Y) = -R_{\eta^{-1}*}L_{\xi\eta^{-1}*}Y$. To show the first of these let c be a smooth curve in M with $c'(0) = X$. Then

$f_{*(\xi,\eta)}X = \frac{d}{dt}|_{t=0}c(t)\eta^{-1} = R_{\eta^{-1}*}X$. To show the second recall that if c and c_1 are curves in a Lie group then $\frac{d}{dt}(c(t)c_1(t)) = R_{c_1(t)*}c'(t) + L_{c(t)*}c_1'(t)$ (for example this follows from the "Leibnitz formula" on page 14 of [14] Vol. 1). If $c_1(t) = c(t)^{-1}$ then $c(t)c_1(t)$ is constant whence $0 = R_{c(t)^{-1}*}c'(t) + L_{c(t)*}\frac{d}{dt}c(t)^{-1}$ *i.e.* $\frac{d}{dt}c(t)^{-1} = -L_{c(t)^{-1}*}R_{c(t)^{-1}*}c'(t)$. Now let c be a smooth curve in N with $c'(0) = Y$. Then using what was just shown and that left and right translation commute,

$$\begin{aligned} f_{*(\xi,\eta)}(0,Y) &= \frac{d}{dt}\bigg|_{t=0} \xi c(t)^{-1} \\ &= L_{\xi*}\frac{d}{dt}\bigg|_{t=0} c(t)^{-1} \\ &= -L_{\xi*}L_{\eta^{-1}*}R_{\eta^{-1}*}Y \\ &= -R_{\eta^{-1}*}L_{\xi\eta^{-1}*}Y \end{aligned}$$

This completes the proof.

2.11 LEMMA. *The kernel of $f_{*(\xi,\eta)}$ is $\{(X, L_{\eta\xi^{-1}*}X) : X \in TM_\xi \cap L_{\xi\eta^{-1}*}TN_\eta\}$ and the image of $f_{(\xi,\eta)*}$ is $R_{\eta^{-1}*}(TM_\xi + L_{\xi\eta^{-1}*}TN_\eta)$. Therefore (ξ,η) is a regular point of f if and only if $TM_\xi + L_{\xi\eta^{-1}*}TN_\eta = TG_\xi$.*

PROOF. See the appendix for the definition of a regular point. This lemma follows directly from the last one.

2.12 LEMMA. *For all $g \in G$ define a function $\pi_g : f^{-1}[g] \to M \cap gN$ by*

$$\pi_g(\xi,\eta) = \xi. \tag{2-14}$$

Then for all $g \in G$, φ_g is a bijection of $M \cap gN$ onto $f^{-1}[g]$ and the inverse of φ_g is π_g. If g is a regular value of f then M and gN intersect transversely and thus $M \cap gN$ is a smooth submanifold of G for almost all $g \in G$. If g is a regular value of f then $\varphi_g : M \cap gN \to f^{-1}[g]$ is a diffeomorphism.

PROOF. That φ_g is a bijection with inverse π_g is left to the reader. If g is a regular value of f and $\xi \in M \cap gN$ then let $\eta \in N$ with $\xi = g\eta$. Thus $g = \xi\eta^{-1} = f(\xi,\eta)$ and as g is a regular value of f using lemma 2.11 in the last line,

$$\begin{aligned} TM_\xi + T(gN)_\xi &= TM_\xi + L_{g*}TN_\eta \\ &= TM_\xi + L_{\xi\eta^{-1}*}TN_\eta \\ &= TG_\xi. \end{aligned}$$

This proves M and gN intersect transversely when g is a regular value of f, and by Sard's theorem (see appendix) almost every $g \in G$ is a regular value of f.

If g is a regular value of f then $f^{-1}[g]$ is an embedded submanifold of $M \times N$ and $M \cap gN$ is a submanifold of G as M and gN intersect transversely. From the definitions of φ_g and π_g it is clear they are both smooth functions and as they are inverse to each other this implies that both are diffeomorphisms. This completes the proof.

2.13 We now compute the Jacobian $(Jf)(\xi,\eta)$ at a regular point (ξ,η) of f. First some notation. Let (ξ,η) be a regular point of f and set

$$n=\dim(G),\quad p=\dim(M),\quad q=\dim(N),\quad k=\dim(\mathrm{Kernel}(f_{*(\xi,\eta)}))$$

Then, using 2.11,

$$k=p+q-n=\dim(TM_\xi\cap L_{\xi\eta^{-1}}TN_\eta).$$

Let $X_1,\dots,X_k$ be an orthonormal basis of $TM_\xi\cap L_{\xi\eta^{-1}*}TN_\eta$. Then, as the metric is left invariant,

$$Y_i=L_{\xi\eta^{-1}*}X_i\qquad 1\le i\le k \tag{2-15}$$

is an orthonormal basis of $L_{\eta\xi^{-1}*}TM_\xi\cap TN_\eta$.

Complete $X_1,\dots,X_k$ to an orthonormal basis $X_1,\dots,X_p$ of TM_ξ and $Y_1,\dots,Y_k$ to an orthonormal basis $Y_1,\dots,Y_q$ of TN_η. From 2.11 it follows that

$$Z_i=\frac{1}{\sqrt2}(X_i,L_{\eta\xi^{-1}*}X_i)=\frac{1}{\sqrt2}(X_i,Y_i)\qquad 1\le i\le k \tag{2-16}$$

is an orthonormal basis of $\mathrm{Kernel}(f_{*(\xi,\eta)})$ and therefore if

$$W_i=\frac{1}{\sqrt2}(X_i,-L_{\eta\xi^{-1}*}X_i)=\frac{1}{\sqrt2}(X_i,-Y_i)\qquad 1\le i\le k \tag{2-17}$$

then the $p+q-n$ vectors

$$W_1,\dots,W_k,\quad (X_{k+1},0),\dots,(X_p,0),\quad (0,Y_{k+1}),\dots,(0,Y_q) \tag{2-18}$$

are an orthonormal basis of $\mathrm{Kernel}(f_{*(\xi,\eta)})^\perp$. Using lemma 2.10

$$\begin{aligned}
f_{*(\xi,\eta)}W_i&=f_{*(\xi,\eta)}\frac{1}{\sqrt2}(X_i,-L_{\eta\xi^{-1}*}X_i)\\
&=\frac{1}{\sqrt2}R_{\eta^{-1}*}X_i+R_{\eta^{-1}*}L_{\xi\eta^{-1}}L_{\eta\xi^{-1}*}X_i\\
&=\frac{1}{\sqrt2}(R_{\eta^{-1}*}X_i+R_{\eta^{-1}*}X_i)\\
&=\sqrt2R_{\eta^{-1}*}X_i\\
f_{*(\xi,\eta)}(X_i,0)&=R_{\eta^{-1}*}X_i\\
f_{*(\xi,\eta)}(0,Y_i)&=-R_{\eta^{-1}*}L_{\xi\eta^{-1}*}Y_i
\end{aligned}$$

Using these formulas in the definition of the Jacobian $Jf(\xi,\eta)$ (see appendix) and the formula (2-8) for the modular function,

$$\begin{aligned}
Jf(\xi,\eta)&=\|f_*W_1\wedge\dots\wedge f_*W_k\wedge f_*(X_{k+1},0)\wedge\cdots\\
&\qquad\wedge f_*(X_p,0)\wedge f_*(0,Y_{k+1})\wedge\dots\wedge(0,Y_q)\|\\
&=2^{\frac k2}\|R_{\eta^{-1}*}X_1\wedge\dots\wedge R_{\eta^{-1}*}X_p\wedge R_{\eta^{-1}*}L_{\xi\eta^{-1}*}Y_{k+1}\wedge\dots\wedge R_{\eta^{-1}*}L_{\xi\eta^{-1}}Y_q\|\\
&=2^{\frac k2}\Delta(\eta)\|X_1\wedge\dots\wedge X_p\wedge L_{\xi\eta^{-1}*}Y_{k+1}\wedge\dots\wedge L_{\xi\eta^{-1}*}Y_q\|\\
&=2^{\frac k2}\|L_{\xi^{-1}*}X_1\wedge\dots\wedge L_{\xi^{-1}*}X_p\wedge L_{\eta^{-1}*}Y^{k+1}\wedge\dots\wedge L_{\eta^{-1}*}Y_q\|
\end{aligned}$$

(2-19)

$$=2^{\frac k2}\|u_1\wedge\dots\wedge u_k\wedge v_{k+1}\wedge\dots\wedge v_p\wedge w_{k+1}\wedge\dots\wedge w_q\|$$

where, to simplify notation, we have set

$$\begin{aligned} u_i &= L_{\xi^{-1}*}X_i = L_{\eta^{-1}*}Y_i && 1 \le i \le k \\ v_i &= L_{\xi^{-1}*}X_i && k+1 \le i \le p \\ w_i &= L_{\eta^{-1}*}Y_i && k+1 \le i \le q \end{aligned}$$

Also set $V = L_{\xi^{-1}}TM_\xi$ and $W = L_{\eta^{-1}*}TN_\eta$. Then from the left invariance of the metric and the definition of the X_i's and Y_i's it follows

$$\begin{array}{rll} u_1, \dots, u_k & \text{is an orthonormal basis of} & V \cap W \\ v_{k+1}, \dots, v_p & \text{is an orthonormal basis of} & (V \cap W)^\perp \cap V \\ w_{k+1}, \dots, w_q & \text{is an orthonormal basis of} & (V \cap W)^\perp \cap W \end{array}$$

Therefore each u_i is orthogonal to each v_j and each w_j whence ($I_a = a \times a$ identity matrix)

$$\begin{aligned} &\|u_1 \wedge \cdots \wedge u_k \wedge v_{k+1} \wedge \cdots \wedge v_p \wedge w_{k+1} \wedge \cdots \wedge_q \|^2 \\ &\qquad = \det \begin{bmatrix} \langle u_i, u_j \rangle & \langle v_i, u_j \rangle & \langle w_i, u_j \rangle \\ \langle u_i, v_j \rangle & \langle v_i, v_j \rangle & \langle w_i, w_j \rangle \\ \langle u_i, w_j \rangle & \langle v_i, w_j \rangle & \langle w_i, w_j \rangle \end{bmatrix} \\ &\qquad = \det \begin{bmatrix} I_k & 0 & 0 \\ 0 & I_{p-k} & \langle v_i, w_j \rangle \\ 0 & \langle w_j, v_i \rangle & I_{q-k} \end{bmatrix} \\ &\qquad = \det \begin{bmatrix} I_{p-k} & \langle v_i, w_j \rangle \\ \langle w_i, v_j \rangle & I_{q-k} \end{bmatrix} \\ &\qquad = \|v_{k+1} \wedge \cdots \wedge v_p \wedge w_{k+1} \wedge \cdots \wedge w_q\|^2 \end{aligned}$$

Using this in equation (2-19) yields

$$(Jf)(\xi, \eta) = 2^{\frac{k}{2}} \Delta(\eta) \|v_{k+1} \wedge \cdots \wedge v_p \wedge w_{k+1} \wedge \cdots \wedge w_k\| \tag{2-20}$$

We still have to relate this to the angle $\sigma(T^\perp M_\xi, T^\perp N_\eta)$. To do this let

$$U = \operatorname{span}\{v_{k+1}, \dots, v_p, w_{k+1}, \dots, w_q\}.$$

This is a vector space of dimension $n-k$. Complete $v_{k+1}, \dots, v_p$ to an orthonormal basis $v_{k+1}, \dots, v_n$ of U. Then $v_{p+1}, \dots, v_n$ is an orthonormal basis of $V^\perp = L_{\xi^{-1}*}T^\perp M_\xi$. Likewise if $w_{k+1}, \dots, w_q$ is completed to an orthonormal basis $w_{k+1}, \dots, w_n$ of U then $w_{q+1}, \dots, w_n$ is a basis of $W^\perp = L_{\eta^{-1}*}T^\perp N_\eta$.

Because the dimension of U is $n-k$ the Hodge star on U maps $\bigwedge^r(U)$ to $\bigwedge^{n-k-r}(U)$ (see the book [11] page 15) and

$$\begin{aligned} v_{k+1} \wedge \cdots \wedge v_p &= \pm * (v_{p+1} \wedge \cdots \wedge v_n) \\ w_{k+1} \wedge \cdots \wedge w_q &= \pm * (w_{q+1} \wedge \cdots \wedge w_n) \end{aligned}$$

Using known identities for $*$ (see page 16 of [11]) and the last two equations,

$$\begin{aligned} v_{k+1} \wedge \cdots \wedge v_p \wedge w_{k+1} \wedge \cdots \wedge w_q \\ &= \pm(v_{k+1} \wedge \cdots \wedge v_p) \wedge *(w_{q+1} \wedge \cdots \wedge w_n) \\ &= \pm(*v_{k+1} \wedge \cdots \wedge v_p) \wedge w_{q+1} \wedge \cdots \wedge w_n \\ &= \pm v_{p+1} \wedge \cdots \wedge v_n \wedge w_{q+1} \wedge \cdots \wedge w_n. \end{aligned}$$

Using this in (2-20) and recalling the definition of $\sigma(T^{\perp}M_\xi, T^{\perp}N_\eta)$,

$$\begin{aligned} Jf(\xi,\eta) &= 2^{\frac{k}{2}}\Delta(\eta)\|v_{p+1} \wedge \cdots \wedge v_n \wedge w_{q+1} \wedge \cdots \wedge w_n\| \\ &= 2^{\frac{k}{2}}\Delta(\eta)\sigma(T^{\perp}M_\xi, T^{\perp}N_\eta). \end{aligned} \tag{2-21}$$

2.14 It remains to relate $\int_{f^{-1}[g]} h\,\Omega_{f^{-1}[g]}$ to $\int_{M\cap gN} h \circ \varphi_g\,\Omega_{M\cap gN}$. Let g be a regular value of f. Then, by 2.12, $\varphi_g : M \cap gN \to f^{-1}[g]$ is a diffeomorphism with inverse π_g. If $(\xi,\eta) \in f^{-1}[g]$ then, using the notation of equation (2-16), $Z_1,\ldots,Z_k$ is an orthonormal basis of Kernel$(f_{*(\xi,\eta)}) = T(f^{-1}[g])_{(\xi,\eta)}$. From the definition of π_g and Z_i it is clear $\pi_{g*}Z_i = \frac{1}{\sqrt{2}}\pi_{g*}(X_i,X_i) = \frac{1}{\sqrt{2}}X_i$. But $X_1,\ldots,X_k$ is an orthonormal basis of $T(M\cap gN)_\xi$ and π_g is the inverse of φ_g. Therefore we have just shown $\varphi_{g*}X_i = \sqrt{2}Z_i$ for $1 < i < k$. This implies $\varphi_g^*\Omega_{f^{-1}[g]} = 2^{k/2}\,\Omega_{M\cap gN}$, so that by the change of variable formula,

$$\int_{f^{-1}[g]} h\,\Omega_{f^{-1}[g]} = 2^{\frac{k}{2}} \int_{M\cap gN} h \circ \varphi_g\,\Omega_{M\cap gN}.$$

Using this equation and equation (2-21) in equation (2-12) yields (2-9) and completes the proof of the basic integral formula.

3. Poincaré's formula in homogeneous spaces.

3.1 In this section G will be a Lie group and K a compact subgroup of G. Let G/K be the homogeneous space of left cosets ξK of K in G. Then G can be viewed as a group of transformations of G/K by letting $g \in G$ send $\xi K \in G/K$ to $g\xi K$. Let $\pi : G \to G/K$ be the natural projection. Then $\pi(e)$ (e is the identity element of G) will be called the origin of G/K a and denoted by "**o**".

It will be assumed that G has a left invariant Riemannian metric $\langle\,,\rangle$ that is also right invariant under elements of K. This metric induces a unique Riemannian metric on G/K, which will also be denoted by "$\langle\,,\rangle$", that makes π into a Riemannian submersion. It can be defined as follows. Let $x \in G/K$ and choose any element $\xi \in G$ with $\pi(\xi) = x$. Then $\pi_{*\xi}$ restricted to Kernel$(\pi_{*\xi})^{\perp}$ is a linear isomorphism of Kernel$(\pi_{*\xi})^{\perp}$ onto $T(G/K)_x$. Define the metric on $T(G/K)_x$ by

$$\langle X,Y\rangle_{T(G/K)_x} = \langle \pi_*|^{-1}_{\ker(\pi_{*\xi})}X,\, \pi_*|^{-1}_{\ker(\pi_{*\xi})}Y\rangle \tag{3-1}$$

The right invariance of the metric on G under elements of K shows that this is independent of the choice of ξ with $\pi(\xi) = x$. This metric on G/K is invariant under G. We remark that for every Riemannian metric on G/K that is invariant under G there is a left invariant metric on G that is also right invariant by$K$ that induces the given metric on G/K in the above manner.

3.2 We would like to be able to define angles between subspaces tangent to G/K at different points as we did in the case of subspaces of tangent spaces to G. In the latter case we left translated both subspaces back to the identity element of G and then found the angle between these subspaces. If V is a subspace of $T(G/K)_x$ and W is a subspace of $T(G/K)_y$ then there are $\xi, \eta \in G$ with $\xi(\mathbf{o}) = x$, $\eta(\mathbf{o}) = y$ and we could try to define the angle between V and W as the angle between $\xi_*^{-1}V$ and $\eta_*^{-1}W$ but this is not well defined as the choice of ξ and η is not unique. This problem can be overcome by averaging over all possible choices of η.

3.3 Definition. *If $x, y \in G/K$ and V is a subspace of $T(G/K)_x$ and W is a subspace of $T(G/K)_y$ then define $\sigma_K(V, W)$ by*

$$\sigma_K(V, W) = \int_K \sigma(\xi_*^{-1}W, a_*^{-1}\eta_*^{-1}W)\, \Omega_K(a)$$

where ξ, η are elements of G with $\xi(\mathbf{o}) = x$, $\eta(\mathbf{o}) = y$ (or what is the same thing $\pi(\xi) = x$, $\pi(\eta) = y$).

3.4 Proposition. *The function $\sigma_K(V, W)$ is independent of the choice of ξ and η and for all $g \in G$ satisfies*

$$\begin{aligned} \sigma_K(V, W) = \sigma_K(W, V) = \sigma_K(g_*V, W) = \sigma_K(V, g_*W), \\ \sigma_K(V, \{0\}) = \mathrm{Vol}(K). \end{aligned} \tag{3-2}$$

Proof. This follows from equation (2-3) and that K is compact (and thus unimodular) so that the measure Ω_K is invariant under the changes of variable $a \mapsto a^{-1}$, $a \mapsto ab$ and $a \mapsto ba$ for fixed $b \in K$. For example if ξ_1 is any other element of G with $\xi_1(\mathbf{o}) = x$ then $\xi_1 = \xi b$ for some $b \in K$. Therefore

$$\begin{aligned} \int_K \sigma(\xi_{1*}^{-1}V, a_*^{-1}\eta_*^{-1}W)\, \Omega_K(a) &= \int_K \sigma(b_*^{-1}\xi_*^{-1}V, a_*^{-1}\eta_*^{-1}W)\, \Omega_K(a) \\ &= \int_K \sigma(\xi_*^{-1}V, b_*a_*^{-1}\eta_*^{-1}W)\, \Omega_K(a) \\ &= \int_K \sigma(\xi_*^{-1}V, a_*^{-1}\eta_*^{-1}W)\, \Omega_K(a) \end{aligned}$$

where the step going from the first to the second line uses equation (2-3) with $\rho = b_*$ and the last step used the invariance under the change of variable $a \mapsto ab$. That $\sigma_K(V, \{0\}) = \mathrm{Vol}(K)$ follows from $\sigma(V, \{0\}) = 1$.

3.5 It is convenient to list one more elementary property of the averaged angle $\sigma_K(V, W)$ that allows angles between pairs of subspaces on one homogeneous space to be related to angles between pairs of subspaces on another homogeneous space. This is a preliminary to the transfer principle of the next section. Let G' be another Lie group and K' a compact subgroup of G' so that G and G' have the same dimension, K and K' have the same dimension. Suppose that G' has a Riemannian metric $\langle\, ,\rangle'$ that is left invariant by G' and right invariant by K'. Give G'/K' the metric that makes the natural projection from G' onto G'/K' a Riemannian submersion. Assume that there is a smooth isomorphism $\rho : K \to K'$ and a linear isometry $\psi : T(G/K)_{\mathbf{o}} \to T(G'/K')_{\mathbf{o}}$ that intertwines ρ, that is

$$\psi \circ a_* = \rho(a)_* \circ \psi$$

for all $a \in K$. Given $x, y \in G/K$, V a subspace of $T(G/K)_x$, W a subspace of $T(G/K)_y$, and $\xi, \eta \in G$ with $\xi(\mathbf{o}) = x$ and $\eta(0) = y$ Also let $x', y', \in G'/K'$, V' a subspace of $T(G'/K')_{x'}$, W' a subspace of $T(G'/K')_{y'}$. Let ξ', η' be elements of G' with $\xi(\mathbf{o}') = x'$ and $\eta(\mathbf{o}') = y'$. Assume that

$$\psi\xi_*^{-1}V = (\xi')_*^{-1}V'$$
$$\psi\eta_*^{-1}W = (\eta')_*^{-1}W'$$
$$\mathrm{Vol}(K) = \mathrm{Vol}(K')$$

Then

$$\sigma_K(V, W) = \sigma'_K(V', W') \tag{3-3}$$

The proof is nothing more than a change of variable in the integral defining $\sigma_K(V, W)$ and is left to the reader.

3.6 The modular function Δ defined in paragraph 2.3 is a smooth homomorphism of G into the multiplicative group of positive real numbers. Therefore $\Delta[K]$ is a compact group and as the only compact subgroup of the positive reals is the group $\{1\}$ it follows that $\Delta(a) = 1$ for all $a \in K$. If $\eta \in G/K$ and $\pi(\eta_1) = \pi(\eta)$ then $\eta_1 = \eta$ for some $a \in K$ and thus $\Delta(\eta_1) = \Delta(\eta a) = \Delta(\eta)\Delta(a) = \Delta(\eta)$ thus the following makes sense.

3.7 Definition. *Let $\Delta_K : G/K \to (0, \infty)$ be given by*

$$\Delta_K(y) = \Delta(\eta) \qquad \textit{where} \qquad \pi(\eta) = y.$$

3.8 Poincaré's formula for homogeneous spaces. *Let M, N be compact submanifolds (possibly with boundary) of G/K with $\dim(M) + \dim(N) \geq \dim(G/K)$. Then for almost all $g \in G$ the submanifolds M and gN intersect transversely and*

$$\int_G \mathrm{Vol}(M \cap gN)\, \Omega_G(g) = \iint_{M\times N} \sigma_K(T^\perp M_x, T^\perp N_y)\Delta_K(y)\, \Omega_{M\times N}(x, y) \tag{3-4}$$

3.9 Corollary. *Under the hypothesis of 3.8:*
(a) If G is transitive on the set of tangent spaces to M then

$$\int_G \mathrm{Vol}(M \cap gN)\, \Omega_G(g) = \mathrm{Vol}(M) \int_N \sigma_K(T^\perp M_{x_0}, T^\perp N_y)\Delta_K(y)\, \Omega(y)$$

where x_0 is any point of M. (The function $y \mapsto \sigma_K(T^\perp M_{x_0}, T^\perp N_y)$ is independent of the choice of x_0 by equation (3-2).)

(b) If in addition to the hypothesis of (a) G is transitive on the set of tangent spaces to N then

$$\int_G \mathrm{Vol}(M \cap gN)\, \Omega_G(g) = \sigma_K(T^\perp M_{x_0}, T^\perp N_{y_0}) \int_N \Delta_K(y)\, \Omega(y)$$

where y_0 is any point of N. (The number $\sigma_K(T^\perp M_{x_0}, T^\perp N_{y_0})$ is independent of the choice of $x_0 \in M$ and $y_0 \in N$).

(c) If in addition to the hypothesis of (a) and (b) G is unimodular then

$$\int_G \mathrm{Vol}(M \cap gN)\, \Omega_G(g) = \sigma_K(T^\perp M_{x_0}, T^\perp N_{y_0})\, \mathrm{Vol}(M)\, \mathrm{Vol}(N) \tag{3-5}$$

where x_0 is any point of M and y_0 any point of N.

Proof of corollary. It follows at once from 3.8 and the transformation rules (3-2) for σ_K.

3.10 Remarks. (1) All the results of the corollary are in section 5 of the paper [2] of Brothers. He does not state the more general result (3.8), however his methods can clearly be modified to cover this case also. His proofs are harder as he proves the results in the case that M and N are normal currents; thus the analysis involved in the proof becomes much more complicated, in particular an entire section of [2] is devoted to the intersection theory of currents in a homogeneous space, something that is trivial in the smooth case covered here.

(2) In section 11 of [2], Brothers gives an interesting generalization of 3.8 that covers the case that $\sigma_K(T^\perp M_x, T^\perp N_x) = 0$ for all $x \in M$ and and $y \in N$. By use of the generalized version of our basic integral formula mentioned in remark (2.8) (2) the methods here can also be used to prove this generalization. In the example of remark (2.8) (2), where G is the group of translations of $\mathbb{R}^2$ and M and N are segments parallel to the x-axis, then Brothers result reduces to

$$\int_G \mathrm{length}(M \cap gN)\, d\mathcal{H}^1(g) = \mathrm{length}(M)\mathrm{length}(N)$$

where $\mathcal{H}^1$ is the one dimensional Hausdorff measure on $G = \mathbb{R}^2$.

3.11 We now give the proof of 3.8. Let $\widehat{M} = \pi^{-1}M$ and $\widehat{N} = \pi^{-1}N$ be the preimages of M and N under the natural projection $\pi : G \to G/K$. Apply the basic integral formula (2.7) to the submanifolds $\widehat{M}$ and $\widehat{N}$ of G with $h \equiv 1$ to get that for almost all $g \in G$ that $\widehat{M}$ and $g\widehat{N}$ intersect transversely and

$$\int_G \mathrm{Vol}(\widehat{M} \cap g\widehat{N})\, \Omega_G(g) = \iint_{\widehat{N} \times \widehat{M}} \Delta(\eta)\sigma(T^\perp \widehat{M}_\xi, T^\perp \widehat{N}_\eta)\, \Omega_{\widehat{M} \times \widehat{N}}(\xi, \eta) \tag{3-6}$$

Because π is a submersion, $\widehat{M} = \pi^{-1}M$ and $g\widehat{N} = g\pi^{-1}N = \pi^{-1}gN$ intersect transversely if and only if M and gN intersect transversely. Thus M and gN

intersect transversely for almost all $g \in G$. If $g \in G$ is so that M and gN intersect transversely then the restriction of π to $\widehat{M} \cap g\widehat{N}$ is a Riemannian submersion of $\widehat{M} \cap g\widehat{N}$ onto $M \cap gN$ that fibers with each fibre isometric to K. Therefore $\mathrm{Vol}(\widehat{M} \cap g\widehat{N}) = \mathrm{Vol}(K)\,\mathrm{Vol}(M \cap gN)$ for almost all $g \in G$ so that

$$\int_G \mathrm{Vol}(\widehat{M} \cap g\widehat{N})\,\Omega_G(g) = \mathrm{Vol}(K) \int_G \mathrm{Vol}(M \cap gN)\,\Omega_G(g) \tag{3-7}$$

We now go to work on the right side of equation (3-6). The map $(\xi, \eta) \mapsto (\pi\xi, \pi\eta)$ from $\widehat{M} \times \widehat{N}$ to $M \times N$ is a Riemannian submersion with fibres $\pi^{-1}[x] \times \pi^{-1}[y]$ that are isometric with $K \times K$. The right side of equation (3-6) can therefore be written as

$$\begin{aligned} &\iint_{\widehat{M}\times\widehat{N}} \Delta(\eta)\sigma(T^\perp\widehat{M}_\xi, T^\perp\widehat{N}_\eta)\,\Omega_{\widehat{M}\times\widehat{N}}(\xi,\eta) \\ &\qquad = \iint_{M\times N} \iint_{\pi^{-1}[x]\times\pi^{-1}[y]} \Delta(\eta)\sigma(T^\perp\widehat{M}_\xi, T^\perp\widehat{N}_\eta)\,\Omega_{\pi^{-1}[x]\times\pi^{-1}[y]}(\xi,\eta)\,\Omega_{M\times N}(x,y) \\ &\qquad = \iint \mathcal{I}(x,y)\,\Omega_{M\times N}(x,y) \end{aligned} \tag{3-8}$$

where

$$\mathcal{I}(x,y) = \iint_{\pi^{-1}[g]\times\pi^{-1}[y]} \Delta(\eta)\sigma(T^\perp\widehat{M}_\xi, T^\perp\widehat{N}_\eta)\,\Omega_{\pi^{-1}[x]\times\pi^{-1}[y]}(\xi,\eta) \tag{3-9}$$

Choose any $(\xi_0, \eta_0) \in \pi^{-1}[x] \times \pi^{-1}[y]$. Then, because the metric on G is right invariant under elements of K, the map $(a, b) \mapsto (\xi_0 a, \eta_0 b)$ is an isometry of $K \times K$ with $\pi^1[x] \times \pi^{-1}[y]$. Therefore we can change variables in the last equation and use that $\Delta(\eta) = \Delta_K(y)$ for all $\eta \in \pi^{-1}[y]$ to get

$$\mathcal{I}(x,y) = \Delta_K(y) \int_K \int_K \sigma(T^\perp\widehat{M}_{\xi_0 a}, T^\perp\widehat{N}_{\eta_0 b})\,\Omega_K(a)\,\Omega_K(b) \tag{3-10}$$

We now work on the integrand in the last equation. Using equation (2-5)

$$\sigma(T^\perp\widehat{M}_{\xi_0 a}, T^\perp\widehat{N}_{\eta_0 b}) = \sigma(L_{(\xi_0 a)^{-1}*}T^\perp\widehat{M}_{\xi_0 a}, L_{(\eta_0 b)^{-1}*}T^\perp\widehat{N}_{\eta_0 b}) \tag{3-11}$$

Both $L_{(\xi_0 a)^{-1}*}T^\perp\widehat{M}_{\xi_0 a}$ and $L_{(\eta_0 b)^{-1}*}T^\perp\widehat{N}_{\eta_0 b}$ are subspaces of $\mathrm{Kernel}(\pi_{*e})^\perp$ and the restriction of π_{*e} to $\mathrm{Kernel}(\pi_{*e})^\perp$ is a linear isometry onto $T(G/K)_{\mathbf{o}}$. Therefore by equation (2-3) (with $\rho = \pi|_{\mathrm{Ker}(\pi_{*e})^\perp}$)

$$\begin{aligned} \sigma(L_{(\xi_0 a)^{-1}*}T^\perp\widehat{M}_{\xi_0 a}, &L_{(\eta_0 b)^{-1}*}T^\perp\widehat{N}_{\eta_0 b}) \\ &= \sigma(\pi_{*e}L_{(\xi_0 a)^{-1}*}T^\perp\widehat{M}_{\xi_0 a}, \pi_{*e}L_{(\eta_0 b)^{-1}*}T^\perp\widehat{N}_{\eta_0 b}) \end{aligned} \tag{3-12}$$

But $\pi L_g = g\pi$ so that

$$\pi_{*e}L_{(\xi_0 a)^{-1}*}T^\perp\widehat{M}_{\xi_0 a} = (\xi_0 a)_*^{-1}\pi_{*\xi_0 a}T^\perp\widehat{M}_{\xi_0 a} = (\xi_0 a)_*^{-1}T^\perp M_x$$

where $\pi_{*\xi_0 a}T^\perp\widehat{M}_{\xi_0 a} = T^\perp M_x$ as π is a Riemannian submersion. Likewise $\pi_{*e}T^\perp\widehat{N}_{\eta_0 b} = (\eta_0 b)_*^{-1}T^\perp N_y$. Using these in (3-12) and the result of that in (3-11) yields

$$\begin{aligned}\sigma(T^\perp\widehat{M}_{\xi_0 a}, T^\perp\widehat{N}_{\eta_0 b}) &= \sigma((\xi_0 a)_*^{-1}T^\perp M_x, (\eta_0 b)_*^{-1}T^\perp\widehat{N}_y)\\ &= \sigma((\xi_0 a)_*^{-1}T^\perp M_x, b_*^{-1}\eta_{0*}^{-1}TN_y)\end{aligned} \tag{3-13}$$

Put this in (3-10) and recall the definition of σ_K to get

$$\begin{aligned}\mathcal{I}(x,y) &= \Delta_K(y)\int_K\int_K \sigma((\xi_0 a)_*^{-1}T^\perp M_x, (b_*^{-1}\eta_{0*}^{-1})T^\perp N_y)\,\Omega_K(b)\,\Omega_K(a)\\ &= \Delta_K(y)\int_K \sigma_K(T^\perp M_x, T^\perp N_y)\,\Omega_K(a)\\ &= \operatorname{Vol}(K)\Delta_K(y)\sigma_K(T^\perp M_x, T^\perp N_y).\end{aligned} \tag{3-14}$$

The equations (3-14), (3-8), (3-7) and (3-6) together imply equation (3-4). This completes the proof of 3.8.

3.12 Examples Here we will show how the constant $\sigma_K(T^\perp M_{x_0}, T^\perp N_{y_0})$ in equation (3-5) can be computed by evaluating the integral $\int_G \operatorname{Vol}(M\cap gN)\,\Omega_G(g)$ for the proper choice of M and N and how equation (3-3) can then be used to transfer the value of this constant to other homogeneous spaces with the same isotropy subgroup K. This is the transfer principle of the next section in the present context. The first two examples are well known, the third seems to be new, the fourth is a lemma that will be used later and the fifth is a proposition about hypersurfaces in two point homogeneous spaces and is rather more sophisticated. In these examples the values of all constants will be expressed in terms of the volumes of the standard spheres S^k (the set of unit vectors in $\mathbb{R}^{k+1}$). These volumes have the well known values

$$\operatorname{Vol}(S^k) = \frac{2(\pi)^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}$$

where Γ is the gamma function.

(a) We start with the case G/K has constant sectional curvature. First consider S^n, which has constant sectional curvature one. The group of orientation preserving isometries of S^n is the matrix group $SO(n{+}1)$ (the group of real orthogonal matrices with determinant $+1$) and the isotropy subgroup of S^n at the north pole is the subgroup $SO(n)$, imbedded in in the natural way. Therefore $SO(n+1)/SO(n) = S^n$. To define a Riemannian metric on $SO(n+1)$, first define an inner product $\langle\,,\rangle$ on the vector space of $(n+1)\times(n+1)$ matrices by $\langle A,B\rangle = \frac{1}{2}\operatorname{trace}(AB^t)$ (B^t is the transpose of B); and then give $SO(n+1)$ the metric it has as a submanifold of this inner product space. This metric is both left and right invariant by elements of $SO(n{+}1)$ and makes the natural projection $\pi : SO(n{+}1)\to S^n$ into a Riemannian submersion. Thus $\operatorname{Vol}(SO(n+1)) = \operatorname{Vol}(SO(n))\operatorname{Vol}(S^n)$. So by induction

$$\operatorname{Vol}(SO(k)) = \operatorname{Vol}(S^1)\operatorname{Vol}(S^2)\cdots\operatorname{Vol}(S^{k-1}). \tag{3-15}$$

Because $SO(n+1)$ is transitive on the set of p planes tangent to S^n and also on the set of q planes tangent to S^n the value of $\sigma_{SO(n)}(V^\perp, W^\perp)$ is the same for any p plane V and q plane W tangent to S^n. This value is easily computed by letting $M = S^p$ (imbedded in S^n as a totally geodesic submanifold, *i.e.* as the intersection of S^n with a $p+1$ dimensional linear subspace of $\mathbb{R}^{n+1}$) and $N = S^q$ in equation (3-5) and noting that $S^p \cap gS^q$ is isometric with S^{p+q-n} for almost all $g \in SO(n+1)$. Thus (3-5) yields

$$\sigma_{SO(n)}(T^\perp M_{x_0}, T^\perp N_{y_0}) = \frac{\mathrm{Vol}(S^{p+q-n})\,\mathrm{Vol}(SO(n+1))}{\mathrm{Vol}(S^p)\,\mathrm{Vol}(S^q)} \tag{3-16}$$

Now let G/K be any simply connected Riemannian manifold of constant sectional curvature c, where c can be positive, negative, or zero and let G be the group of orientation preserving isometries of G/K. Then K is smoothly isomorphic with $SO(n)$ and we assume that the volume of K is normalized so that $\mathrm{Vol}(K) = \mathrm{Vol}(SO(n))$. Then we can use equation (3-3) to conclude that if M^p is any compact p dimensional submanifold of G/K and N^q is any compact q dimensional submanifold of G/K, $x_0 \in M$, $y_0 \in N$ that $\sigma(T^\perp M_{x_0}, T^\perp N_{y_0})$ is given by the value on the right side of (3-16). Thus from (3-5) it follows

$$\int_G \mathrm{Vol}(M^p \cap gN^q)\,\Omega_G(g) = \frac{\mathrm{Vol}(S^{p+q-n})\,\mathrm{Vol}(SO(n+1))}{\mathrm{Vol}(S^p)\,\mathrm{Vol}(S^q)}\,\mathrm{Vol}(M^p)\,\mathrm{Vol}(N^q)$$

In particular, this holds when M and N are compact submanifolds of Euclidean space. See the book [18] of Santaló, paragraph 15.2, for another derivation of this formula.

(b) This time we consider complex analytic submanifolds of Kaehler manifolds of constant holomorphic sectional curvature. To begin $\mathbb{CP}^n$ let be the complex projective space of n complex (and $2n$ real) dimensionals. Then the group $U(n+1)$ (the group of $(n+1)$ by $(n+1)$ complex unitary matrices) acts on $\mathbb{CP}^n$ in a natural way. The stabilizer of a point of $\mathbb{CP}^n$ is then $U(1) \times U(n)$ imbedded in $U(n+1)$ in the natural way. Thus $\mathbb{CP}^n$ can be realized as a homogeneous space as $\mathbb{CP}^n = U(n+1)/(U(1) \times U(n))$. Put a Riemannian metric on $U(n+1)$ by first putting a real inner product $\langle\,,\rangle$ on the $(n+1)$ by $(n+1)$ complex matrices by

$$\langle A, B\rangle = \frac{1}{2}\text{real part of trace}(AB^*)$$

(where B^* = conjugate transpose of B) and giving $U(n+1)$ the metric induced on it as a submanifold of this inner product space. This metric is invariant under both left and right translations by elements of $U(n+1)$. Give $\mathbb{CP}^n$ the metric that makes the natural projection $\pi : U(n+1) \to U(n+1)/(U(1) \times U(n)) = \mathbb{CP}^n$ into a Riemannian submersion. (For details of the construction just outlined see volume 2 of [14] example 10.5 on pages 273-278.) With this metric $\mathbb{CP}^n$ is a Kaehler manifold such that all the holomorphic sectional curvatures are 4 and all the totally real sectional curvatures are 1. There is a Riemannian submersion of S^{2k+1} onto $\mathbb{CP}^k$ (the Hopf fibration) that fibers with fibre S^1. Thus

$$\mathrm{Vol}(\mathbb{CP}^k) = \frac{1}{2\pi}\,\mathrm{Vol}(S^{2k+1})$$

Considering S^k as the set of unit vectors in $\mathbb{C}^{k+1}$ we see that $U(k+1)$ acts transitively on S^{2k+1} and that the stabilizer in $U(k+1)$ of a point of S^{2k+1} is conjugate to $U(k)$. Thus $S^{2k+1} = U(k+1)/U(k)$ and the natural projection induced from $U(k+1)$ to S^{2k+1} is a Riemannian submersion. Therefore $\mathrm{Vol}(U(k+1)) = \mathrm{Vol}(S^{2k+1})\,\mathrm{Vol}(U(k))$ and whence

$$\mathrm{Vol}(U(n+1)) = \mathrm{Vol}(S^{2n+1})\,\mathrm{Vol}(S^{2n-1})\cdots\mathrm{Vol}(S^3)\,\mathrm{Vol}(S^1).$$

If M^p is any complex submanifold of $\mathbb{CP}^n$ of complex dimension p and N^q is any complex submanifold of complex dimension q then the number $\sigma_{U(1)\times U(n)}(T^\perp M^p_{x_0}, T^\perp N^q_{y_0})$ with $x_0 \in M^p$ and $y_0 \in N^q$ is independent of x_0, y_0, M and N. Therefore it can be computed from equation (3-5) by letting $M^p = \mathbb{CP}^p$, $N^q = \mathbb{CP}^q$, and noting that in this case $M^p \cap gN^q = \mathbb{CP}^{p+q-n}$ for almost all $g \in U(n+1)$. This yields

$$\sigma_{U(1)\times U(n)}(T^\perp M^q_{x_0}, T^\perp N^q_{y_0}) = \frac{\mathrm{Vol}(\mathbb{CP}^{p+q-n})\,\mathrm{Vol}(U(n+1))}{\mathrm{Vol}(\mathbb{CP}^p)\,\mathrm{Vol}(\mathbb{CP}^q)} \tag{3-17}$$

Now let E be any simply connected Kaehler manifold of constant holomorphic sectional curvature c and complex dimension n. Then E can be realized as a homogeneous space G/K where K is smoothly isometric with $U(1)\times U(n)$ and G acts on E by Kaehler isometries. In the case c is positive G is isomorphic with $U(n+1)$ and in the case c is negative G is isomorphic with $U(1,n)$. Normalize the metric on K so that $\mathrm{Vol}(K) = \mathrm{Vol}(U(1)\times U(n))$. Let M^p be any compact complex submanifold (possibly with boundary) of complex dimension p and N^q a compact complex submanifold (also possibly with boundary) of complex dimension q. Then by (3-3) the number $\sigma_K(T^\perp M^p_{x_0}, T^\perp N^q_{y_0})$ with $x_0 \in M$ and $y_0 \in N$ is given by the right side of (3-17). Therefore (3-5) yields

$$\int_G \mathrm{Vol}(M^p \cap gN^q)\,\Omega_G(g) = \frac{\mathrm{Vol}(\mathbb{CP}^{p+q-n})\,\mathrm{Vol}(U(n+1))}{\mathrm{Vol}(\mathbb{CP}^p)\,\mathrm{Vol}(\mathbb{CP}^q)}\,\mathrm{Vol}(M^p)\,\mathrm{Vol}(N^q)$$

(c) In this example we again let E be the simply connected Kaehler manifold of constant holomorphic curvature c and complex dimension n; we realize E as a homogeneous space G/K just as before. Then let M^p be a totally real (see [24] for the definition) submanifold of E of real dimension p and N^q a complex submanifold of complex dimension q where $p+2q \geq 2n$. If M and N are compact (possibly with boundary) then

$$\int_G \mathrm{Vol}(M^p \cap gN^q)\,\Omega_G(g) = \int_G \mathrm{Vol}(N^q \cap gM^p)\,\Omega_G(g)$$
$$= \frac{\mathrm{Vol}(\mathbb{RP}^{p+2q-2n})\,\mathrm{Vol}(U(n+1))}{\mathrm{Vol}(\mathbb{RP}^p)\,\mathrm{Vol}(\mathbb{CP}^q)}\,\mathrm{Vol}(M^p)\,\mathrm{Vol}(N^q).$$

where $\mathbb{RP}^k$ is a real projective space with its metric of constant sectional curvature one. It is double covered by S^k, thus $\mathrm{Vol}(\mathbb{RP}^k) = 1/2\,\mathrm{Vol}(S^k)$. In the case $E = G/K = \mathbb{CP}^n$ the above formula is proven by letting $M^p = \mathbb{RP}^p$ imbedded in $\mathbb{CP}^n$ as a totally real and totally geodesic submanifold of $\mathbb{CP}^n$, $N^q = \mathbb{CP}^q$ imbedded as a totally geodesic submanifold, verifying that $M \cap gN$ is isometric with

$\mathbb{RP}^{p+2q-2n}$ for almost all $g \in U(n+1)$ and using this in equation (3-5) to compute $\sigma_K(T^\perp M_{x_0}, T^\perp N_{y_0})$. The details follow the last two examples exactly and are left to the reader.

(d) If in 3.8 we assume that $\dim(N) = \dim(G/K)$, that is, N is the closure of an open set with smooth boundary in G/K, then $T^\perp N_y = \{0\}$ for all $y \in N_0$ whence, by equation (3-2), $\sigma_K(T^\perp M_x, T^\perp N_y) = \mathrm{Vol}(K)$. Therefore equation (3-4) yields

$$\int_G \mathrm{Vol}(M \cap gN)\, \Omega_G(g) = \mathrm{Vol}(K) \int_K \Delta_K(y)\, \Omega_{G/K}(y)\, \mathrm{Vol}(M)$$

If G is unimodular the right side of this equation reduces to $\mathrm{Vol}(K)\,\mathrm{Vol}(N)\,\mathrm{Vol}(M)$.

(e) We now give a less trivial application of the duality principle. Recall that a Riemannian homogeneous space G/K is a **two point homogeneous space** if and only if the action of K on the unit sphere of $T(G/K)_{\mathbf{o}}$ is transitive. This easily implies that G is transitive on the set of tangent spaces to any hypersurface in G/K. The two point homogeneous spaces have been classified ([26] page 295) and in all cases the group G is unimodular.

Proposition. *Let G/K be a two point homogeneous space of dimension n. Let M^p be a p dimensional submanifold of G/K and let N^{n-1} a hypersurface of G/K. If M^p and N^{n-1} have finite volume then*

$$\int_G \mathrm{Vol}(M^p \cap N^{n-1})\, \Omega_G(g) = \frac{\mathrm{Vol}(K)\,\mathrm{Vol}(S^k)\,\mathrm{Vol}(S^n)}{\mathrm{Vol}(S^p)\,\mathrm{Vol}(S^{n-1})}\, \mathrm{Vol}(M^p)\,\mathrm{Vol}(N^{n-1})$$

Remark. This result is somewhat surprising as in most cases G will not be transitive on the set of tangent spaces to M^p.

Proof. Identify $\mathbb{R}^n$ with the tangent space $T(G/K)_{\mathbf{o}}$ of G/K at $\mathbf{o}$. Let $K \rtimes \mathbb{R}^n$ be $K \times \mathbb{R}^n$ with the product Riemannian metric and view it as a group of transformation on $\mathbb{R}^n$ by the rule $(a, v)X = a_* X + v$. The group $K \rtimes \mathbb{R}^n$ then acts on $\mathbb{R}^n$ by isometries. Let V be any p dimensional subspace of $\mathbb{R}^n$ at $0 = \mathbf{o}$ and let B^p be the unit ball in V. Then the translations of R^n, and thus also $K \rtimes \mathbb{R}^n$, is transitive on the set of tangent spaces to B^p. View S^{n-1} as the unit sphere of $\mathbb{R}^n = T(G/K)_{\mathbf{o}}$. Because G/K is a two point homogeneous space the group $K \rtimes \mathbb{R}^n$ is transitive on the set of tangent spaces to S^{n-1}. Note that with the obvious notation $SO(n) \rtimes \mathbb{R}^n$ is the group of orientation preserving isometries of $\mathbb{R}^n$ and thus the results of example (a) apply to this group. By corollary 3.9(c) example (a), obvious symmetry properties of the sphere S^{n-1} and that $\mathrm{Vol}(SO(n+1)) = \mathrm{Vol}(S^n)\,\mathrm{Vol}(SO(n))$ for any $y_0 \in S^{n-1}$

$$\begin{aligned}
\sigma_K(V^\perp, T^\perp(S^{n-1})_{y_0})\,\mathrm{Vol}(B^p)\,\mathrm{Vol}(S^{n-1}) &= \int_{\mathbb{R}^n}\int_K \mathrm{Vol}(B^p \cap (a_* S^{n-1} + v))\, \Omega_K(a)\, \Omega_{\mathbb{R}^n}(v) \\
&= \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(SO(n))} \int_{\mathbb{R}^n}\int_{SO(n)} \mathrm{Vol}(B^p \cap (b_* S^{n-1} + v))\, \Omega_{SO(n)}(b)\, \Omega_{\mathbb{R}^n}(v) \\
&= \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(SO(n))} \frac{\mathrm{Vol}(S^{p-1})\,\mathrm{Vol}(SO(n+1))}{\mathrm{Vol}(S^p)\,\mathrm{Vol}(S^{n-1})}\, \mathrm{Vol}(B^p)\,\mathrm{Vol}(S^{n-1}) \\
&= \frac{\mathrm{Vol}(K)\,\mathrm{Vol}(S^{p-1})\,\mathrm{Vol}(S^n)\,\mathrm{Vol}(B^p)}{\mathrm{Vol}(S^p)}
\end{aligned}$$

So that

$$\sigma_K(V^\perp, T^\perp(S^{n-1})_{y_0}) = \frac{\operatorname{Vol}(K)\operatorname{Vol}(S^{p-1})\operatorname{Vol}(S^n)}{\operatorname{Vol}(S^p)\operatorname{Vol}(S^{n-1})}.$$

Now let M^p, N^{n-1} be as in the theorem, and let $x \in M^p$, $y \in N^{n-1}$. Because V was an arbitrary p dimensional subspace of $\mathbb{R}^n = T(G/K)_{\mathbf{o}}$ it follows from equation (3-3) that $\sigma_K(T^\perp M^p_x, T^\perp N^{n-1}_y)$ is given by the right side of the last equation. The Proposition now follows from Poincaré's formula (3-4). This completes the proof.

Appendix to Section 3: Cauchy-Crofton type formulas and invariant volumes.

3.13 In this appendix the Poincaré formula 3.8 will be used to extend the Cauchy-Crofton formula in Brother's paper [2] from submanifolds M of Riemannian homogeneous spaces G/K for which the group G is transitive on the set tangent spaces to M to arbitrary submanifolds. This formula is of interest not only for its own sake but also because it throws light on the "p-dimensional area" or p dimensional volumes used in the foundational paper [4] of Chern. In his paper Chern proved a Crofton type formula that relates the p dimensional volume of a submanifold to its "average" number of intersections with a "moving plane" (see Chern's paper for details; a brief statement of some of his results is given below.) This p dimensional volume is not defined in terms of an invariant Riemannian metric (Chern does not assume the space G/K has a metric) but is defined "by the method of moving frames of Cartan". In his review of Chern's paper Andre Weil [20] points out that in many homogeneous spaces that there are several distinct ways to define the p dimensional volume so that some clarification is needed. In the case that G/K does have an invariant Riemannian metric it is possible to combine the result proven here with Chern's results to give an explicit formula for Chern's p dimensional volume in terms of the Riemannian data involved. Once this is done it is possible to give examples of (1) submanifolds of a Riemannian homogeneous space where the p dimensional volume of a submanifold in the sense of Chern is different from its Riemannian volume. (2) Distinct p dimensional volumes that both lead to correct Crofton formulas (for different choices of the "moving plane").

In particular we will show there are three notions of two dimensional area for two dimensional submanifolds of $\mathbb{CP}^2$ that are invariant under $U(3)$, only one of which is the usual Riemannian area, and that all three lead to a Crofton type formula. (However for the most interesting set of surfaces in $\mathbb{CP}^2$ the complex curves, the three only differ by a constant factor.) This shows that when constructing the moving frames on the submanifold used to define the invariant p dimensional volume these frames must not only be adapted to the submanifold but also to the type of integral geometric formula that is to be proven.

3.14 The classical formula of Crofton computes the length of a curve in the plane from its average number of points of intersection with a moving line (see [18] for details). We will show that this type of formula can be reduced to the Poincaré formula of 3.8 in a straight forward manner. Many of the details will be left to the reader. In particular the verification that various transversality statements hold almost everywhere will be the readers task. Let G, K, G/K etc. be as in paragraph 3.1.

3.15 Definition. *If S is any subset of G/K let $G(S)$ be the stabilizer of S in G, that is*

$$G(S) = \{g : gS = S\}.$$

3.16 For the rest of this section we will fix a submanifold L_0 of G/K and make the following assumptions and normalizations:

(I) L_0 is a closed imbedded submanifold of G/K of dimension q,
(II) $G(L_0)$ is transitive on L_0,
(III) $\mathbf{o} \in L_0$ ($\mathbf{o} = \pi(e)$ is the origin of G/K) and set $W_0 = T(L_0)_{\mathbf{o}}$.
(IV) The homogeneous space $G/G(L_{\mathbf{o}})$ has a G invariant measure $\Omega_{G/G(L_0)}$

3.17 Remarks. (1) The homogeneous space $G/G(L_0)$ parameterizes the set of all subsets of G/K of the form gL_0 with $g \in G$. For example, if G is the group of isometries of $\mathbb{R}^n$, $G/K = \mathbb{R}^n$ and L_0 is a q dimensional linear subspace of $\mathbb{R}^n$ then $G/G(L_0)$ is the Grassmann manifold of all affine q planes in R^n. In general the analogy between $G/G(L_0)$ and a Grassmann manifold is good, at least with respect to the type of integral geometric formulas that arise. This analogy becomes better in the case that L_0 is totally geodesic.

(2) It follows from (II) that $G(L_0)$ and thus G is transitive on the set of tangent spaces to L_0.

(3) The measure $\Omega_{G/G(L_0)}$, when it exists, is unique up to a positive multiple.

3.18 Our object is to evaluate the integral $\int_{G/G(L_0)} \mathrm{Vol}(M \cap L)\, \Omega_{G/G(L_0)}(L)$ where M is a compact submanifold of G/K with $\dim(M) + \dim(L_0) \geq \dim(G/K)$. To start with, recall that there is a positive constant c_1 (depending only on the choice of the measure $\Omega_{G/G(L_0)}$) such that for all integrable functions h on G

$$\int_{G/G(L_0)} \int_{\pi_0^{-1}[L]} h(g)\, \Omega_{\pi_0^{-1}[L]}(g)\, \Omega_{G/G(L_0)}(L) = c_1 \int_G h(g)\, \Omega_G(g) \tag{3-18}$$

where $\pi_0 G \to G/G(L_0)$ is the natural projection. For a proof of this see §33 of the book [15] of Loomis. For each $L \in G/G(L_0)$ choose a $\xi_L \in G$ with with $\xi_L L_0 = L$. Then, by the left invariance of the metric on G, the map $a \mapsto \xi_L a$ is an isometry of $\pi_0^{-1}[L_0] = G(L_0)$ with $\pi_0^{-1}[L] = \xi_L(L_0)$. Therefore a change of variable in the inner integral on the left side of the last equation leads to

$$\int_{G/G(L_0)} \int_{G(L_0)} h(\xi_L a)\, \Omega_{G(L_0)}(a)\, \Omega_{G/G(L_0)}(L) = c_1 \int_G h(g)\, \Omega_G(g) \tag{3-19}$$

Let M be any compact p dimensional submanifold (possibly with boundary) with $p + q \geq n$ (here $q = \dim(L_0)$) and let N_0 be any open subset of L_0 with smooth boundary and compact closure. If L_0 is compact choose $N_0 = L_0$. Set $h(g) = \mathrm{Vol}(M \cap gN_0)$ in (3-19). Because G is transitive on the set of tangent spaces to L_0 it is also transitive on the set of tangent spaces to N_0. Therefore equation (3-4) allows us to conclude

$$\int h(g)\, \Omega_G(g) = \int_{N_0} \Delta_K(y)\, \Omega_{L_0}(y) \int_M \sigma_K(T^\perp M_x, W_0^\perp)\, \Omega_M(x) \tag{3-20}$$

where $W_0^\perp = T^\perp(L_0)_{\mathbf{o}}$.

For almost all $L \in G/G(L_0)$ the submanifolds L and M intersect transversely so that $M \cap L$ is a $p+q-n$ dimensional submanifold of L. In this case let $a \in G(L_0)$. Then ξ_L induces an isometry of L_0 with L and $\xi_L a N_0 = \xi_L a N_0 \cap L$ (as $\xi_L a N_0 \subseteq L$) whence

$$\begin{aligned} h(\xi_L a) = \operatorname{Vol}(M \cap (\xi_L a) N_0) \qquad &= \operatorname{Vol}(M \cap L \cap \xi_L a N_0) \\ = \operatorname{Vol}(\xi_L^{-1} M \cap \xi_L^{-1} L \cap a N_0) &= \operatorname{Vol}((\xi_L^{-1} M \cap L_0) \cap N_0) \end{aligned}$$

Let $K(L_0) = \{g \in G : g(\mathbf{o}) = \mathbf{o}\} = G(L_0) \cap K$ and let Δ_0 be the modular function of $G(L_0)$. Then Δ_0 induces a function $\Delta_{K(L_0)}$ on $L_0 = G(L_0)/K(L_0)$ just as Δ induced Δ_K on G/K in §3.7. We now apply the results of example 3.12(d) to the submanifolds $(\xi_L^{-1} M \cap L_0)$ and N_0 of the homogeneous space $L_0 = G(L_0)/K_0$ and use the last equation to get

$$\begin{aligned} \int_{G(L_0)} h(\xi_L a)\, \Omega_{G(L_0)}(a) &= \int_{G(L_0)} \operatorname{Vol}((\xi_L^{-1} M \cap L_0) \cap a N_0)\, \Omega_{G(L_0)}(a) \\ &= \operatorname{Vol}(K(L_0)) \int_{N_0} \Delta_{K(L_0)}(y)\, \Omega_{L_0}(y) \operatorname{Vol}(\xi_L^{-1} M \cap L_0) \\ &= c_2 \operatorname{Vol}(M \cap \xi_L L_0) \\ &= c_2 \operatorname{Vol}(M \cap L) \end{aligned}$$

where c_2 denotes the obvious constant. Putting this and (3-20) into (3-19)

$$\int_{G/G(L_0)} \operatorname{Vol}(M \cap L)\, \Omega_{G/G(L_0)}(L) = c \int_M \sigma_K(T^\perp M_x, W_0^\perp)\, \Omega_M(x) \tag{3-22}$$

where

$$c = \frac{\int_{N_0} \Delta_K(y)\, \Omega_{N_0}(y)}{\operatorname{Vol}(K(L_0)) \int_{N_0} \Delta_{K(L_0)}(y)\, \Omega_{L_0}(y)} c_1$$

If both G and $G(L_0)$ are unimodular then this reduces to $c = c_1/(\operatorname{Vol}(K(L_0)))$. Equation (3-21) is the Cauchy-Crofton formula for Riemannian homogeneous spaces. If G is transitive on the set of tangent spaces to M this becomes

$$\int_{G/G(L_0)} \operatorname{Vol}(M \cap L)\, \Omega_{G/G(L_0)}(L) = c\sigma_K(T^\perp M_{x_0}, W_0^\perp) \operatorname{Vol}(M)$$

where x_0 is any point of M. This last equation is due to Brothers (section 7 of [2]). His methods are quite different from those used here and involve a good deal of analysis. In the case that L_0 is compact then (3-21) can be directly related to the Poincaré's formula (3-4) by

$$\begin{aligned} \int_{G/G(L_0)} \operatorname{Vol}(M \cap L)\, \Omega_{G/G(L_0)}(L) &= c \int_M \sigma_K(T^\perp M_x, W_0^\perp)\, \Omega_M(x) \\ &= \frac{c}{\int_{L_0} \Delta_K(y)\, \Omega_{L_0}(y)} \int_G \operatorname{Vol}(M \cap g L_0)\, \Omega(g) \end{aligned} \tag{3-23}$$

3.19 We now consider the Crofton's formula in Chern's paper [4] and its relation to the result proven in the last paragraph. Using the same notation as before we will only consider submanifolds M of G/K of dimension $p = n-q$. Thus if $L \in G/G(L_0)$ intersects M transversely then $M \cap L$ is discrete, so if M is compact $M \cap L$ is finite and $\mathrm{Vol}(M \cap L) = \#(M \cap L)$, the number of points in $M \cap L$. Chern does not assume that G/K has a Riemannian metric invariant under G, or even that K is compact, but instead uses moving frames to construct on each (or at least most) p dimensional submanifolds M a canonically defined p form. Then the p dimensional volume $\mathcal{V}_0(M)$ of M is defined to be the integral over M of this form (see section 3 of his paper). This volume has the property that all $g \in G$ and p dimensional submanifolds M of G/K for which $\mathcal{V}_0(M)$ defined that $\mathcal{V}_0(gM) = \mathcal{V}_0(M)$. Thus this volume is invariant. Chern then proves the Crofton formula

$$\int_{G/G(L_0)} \#(M \cap L)\, \Omega_{G/G(L_0)}(L) = C_{\mathcal{V}_0} \mathcal{V}_0(M) \tag{3-24}$$

where $C_{\mathcal{V}_0}$ is a constant that only depends on the choice of the invariant p dimensional volume $\mathcal{V}_0$. (It should be remarked that Chern does not choose a submanifold L_0 and consider the homogeneous space $G/G(L_0)$, but rather starts with any closed subgroup H of G of codimension q and defines a notation of incidence between elements of G/K and G/H. Thus his formula is much more general than the one just given.) The construction of the volume $\mathcal{V}_0$ is rather subtle therefore it would be nice to have an explicit formula for it in the case that G/K does have an invariant Riemannian metric. Combining equations (3-24) and (3-22) yields required formula

$$\mathcal{V}_0(M) = \frac{c}{C_{\mathcal{V}_0}} \int_M \sigma_K(T^\perp M_x, W_0^\perp)\, \Omega_M(x) \tag{3-25}$$

This expression for $\mathcal{V}_0(M)$ shows at once that if the map $V \mapsto \sigma_K(V, W_0^\perp)$ defined on the set of all p dimensional linear subspaces of $T(G/K)_{\mathbf{o}}$ is not constant then $\mathcal{V}_0(M)$ is distinct from the Riemannian volume of M. However if V_0 is a p dimensional subspace of $T(G/K)_{\mathbf{o}}$ and we restrict our attention to submanifolds M of G/K of type V_0 (as defined in the next section) then $\mathcal{V}_0$ is just a constant (depending on V_0) times the Riemannian volume.

The formula also shows that $\mathcal{V}_0$ depends not only on just the dimension p of the submanifold M but also on $W_0 = T(L_0)_{\mathbf{o}}$. This raises the question: If L_1 is another submanifold of G/K of the same dimension q as L_0 and L_1 satisfies the conditions (I), (II), (III) and (IV) given in 3.16 does the Crofton formula

$$\int_{G/G(L_1)} \#(M \cap L)\, \Omega_{G/G(L_1)}(L) = (\text{const.}) \mathcal{V}_0 \tag{3-26}$$

hold for all p dimensional submanifolds M of G/K? The answer is "no" as we now show. If it did hold then it would imply that

$$\int \sigma_K(T^\perp M_x, W_1^\perp)\, \Omega_M(x) = (\text{const.}) \int_M \sigma_K(T^\perp M_x, W_0^\perp)\, \Omega_M(x)$$

(with $W_1 = T(L_1)_{\mathbf{o}}$) for all p dimensional M. This in turn gives

$$\sigma_K(V, W_1^\perp) = (\text{const.})\sigma_K(V, W_0^\perp) \tag{3-27}$$

for all p dimensional subspaces V of $T(G/K)_{\mathbf{o}}$ as a necessary and sufficient condition for (3-26) to hold.

The simplest examples where this fails can be constructed by letting $K = \{e\}$ so that $G/K = G$. In this case in it is not hard to show that (3-27) holds if and only if $W_1 = W_0$. Thus we only have to choose L_0 and L_1 with $T(L_0)_{\mathbf{o}} = W_0 \neq W_1 = T(L_1)_{\mathbf{o}}$ to get a counterexample. It is also worth noting that in this case Chern's invariant volume $\mathcal{V}_0(M)$ is the integral over M of a left invariant p form defined on all of $G/K = G$; on more complicated homogeneous spaces this is not the case.

These last examples are not very satisfying as the spaces where we do most of our geometry have large isotropy subgroup. Thus let $G/K = U(n+1)/(U(1)\times U(n)) = \mathbb{CP}^n$ with the metric defined in example 3.10(b) and assume that $n = 2k$ is even. Let $L_0 = \mathbb{CP}^k$ imbedded in $\mathbb{CP}^{2k}$ in the usual manner and let $L_1 = \mathbb{RP}^{2k}$ imbedded as a totally real totally geodesic submanifold of $\mathbb{CP}^{2k}$. Let x_0 be any point of $L_0 = \mathbb{CP}^k$ and x_1 any point of $L_1 = \mathbb{RP}^{2k}$. Then (3-27) is equivalent to

$$\sigma_K(V, T^\perp \mathbb{RP}^{2k}_{x_1}) = \gamma \sigma_K(V, T^\perp \mathbb{CP}^k_{x_0}) \qquad (K = U(1)\times U(n))$$

for some constant γ and all real linear subspaces V of $T(\mathbb{CP}^{2k})_{\mathbf{o}}$ of real dimension $2k$. Putting this in equation (3-4) (and using that $\Delta_H \equiv 1$ in this case) yields that

$$\int_{U(2k+1)} \#(M\cap g\mathbb{RP}^{2k})\,\Omega_{U(2k+1)} = \gamma \int_{U(2k+1)} \#(M\cap g\mathbb{CP}^k)\,\Omega(g)$$

for all compact submanifolds M of $\mathbb{CP}^{2k}$ of real dimension $2k$. If $M = \mathbb{CP}^k$ then $\#(M\cap g\mathbb{RP}^{2k}) = 1$ and $\#(M\cap g\mathbb{CP}^k) = 1$ for almost all $g \in U(2k+1)$ so the last equation gives $\gamma = 1$. On the other hand, if $M = \mathbb{RP}^{2k}$ then $\#(M\cap g\mathbb{RP}^{2k}) = 2k+1$ and $\#(M\cap g\mathbb{CP}^k) = 1$ for almost all g which would give $\gamma = 2k+1$. (See the following paragraph for the computation of these intersection numbers.) This contradiction shows that (3-26) cannot hold.

Therefore there are two invariant volumes $\mathcal{V}_0$, $\mathcal{V}_1$ given by

$$\mathcal{V}_0(M) = \int_M \sigma_K(T^\perp M_x, T^\perp(\mathbb{CP}^k)_{x_0})\,\Omega_M(x)$$

and

$$\mathcal{V}_1(M) = \int_M \sigma_K(T^\perp M_x, T^\perp(\mathbb{RP}^{2k})_{x_1})\,\Omega_M(x)$$

in addition to the Riemannian volume of M and all three have integral geometric meaning *via* one of the various Crofton formulas.

We now compute the intersection numbers used above. Let M and N be totally geodesic submanifolds of $\mathbb{CP}^n$ with $\dim(M) + \dim(N) = n$ and assume M and N intersect transversely. Then $M\cap N$ is a finite subset of $\mathbb{CP}^n$. Because $\mathbb{CP}^n$ has positive sectional curvatures (see [27]) a theorem of Frankel [28] implies $M \cap N$ contains at least one point x. Let $C(x)$ be the cut locus of x in $\mathbb{CP}^n$ (see [27] for definition). Then any other point of $M \cap N$ is in $C(x)$. (For if $x \neq y \in M \cap N$, $y \notin C(x)$ then there is a unique length minimizing geodesic segment $[xy]$ joining x to y. But M and N are totally geodesic so this would imply that $[xy] \subset M \cap N$ contradicting that $M \cap N$ is finite.) Therefore

$$\#(M \cap N) = 1 + \#((M \cap C(x)) \cap (N \cap C(x))) \tag{3-28}$$

But (see [27]) $C(x)$ is isometric to $\mathbb{CP}^{n-1}$ and is imbedded as a totally geodesic submanifold. Also $M \cap C(x)$ and $N \cap C(x)$ intersect transversely in $C(x)$ and $\dim(N \cap C(x)) \leq \dim(N) - 1$. If M is a totally geodesic $\mathbb{CP}^k$ in $\mathbb{CP}^n$ then $M \cap C(x)$ is a totally geodesic $\mathbb{CP}^{k-2}$ in $C(x)$ and so $\dim(M \cap C(x)) = \dim(M) - 2$. Thus

$$\dim(M \cap C(x)) + \dim(N \cap C(x)) \leq \dim(M) + \dim(N) - 3 < \dim(C(x)).$$

Therefore the only way $M \cap C(x)$ and $N \cap C(x)$ can have transverse intersection in $C(x)$ is if the intersection is empty. Thus if M or N is isometric with $\mathbb{CP}^k$ then (3-28) implies $\#(M \cap N) = 1$.

If M and N are both isometric to $\mathbb{RP}^n$ then $M \cap C(x)$ and $N \cap C(x)$ are both totally geodesic copies of $\mathbb{RP}^{n-1}$ in $C(x) = \mathbb{CP}^{n-1}$. If $n = 1$ then $\mathbb{CP}^1$ is the Riemann sphere and a copy of $\mathbb{RP}^1$ in $\mathbb{CP}^1$ is a great circle (i.e. a geodesic). Two distinct geodesics intersect in two points. So when $n = 1$, $\#(M \cap N) = 2$. An induction using (3-28) now shows that if M and N are totally geodesic copies of $\mathbb{RP}^n$ in $\mathbb{CP}^n$ which intersect transversely then $\#(M \cap N) = n + 1$.

4. Integral Invariants of Submanifolds of Homogeneous Spaces, the Kinematic Formula and the Transfer Principle

4.1 In this section G, K, G/K and the Riemannian metrics on these spaces will be as described in paragraph 3.1. A very general class of integral invariants of compact submanifolds of the homogeneous space G/K will now be given. Loosely these will be integrals over the submanifold of polynomials in the components of the second fundamental form of the submanifold where, for this to be well defined, the polynomial must be invariant under the isotropy subgroup K in an appropriate sense. In making this definition it is useful to distinguish the case where G is transitive on the set of tangent spaces to the submanifold from the general case.

4.2 Definition. *Let V_0 be a p dimensional subspace of $T(G/K)_{\mathbf{o}}$. Then a p dimensional submanifold M of G/K is of type V_0 if and only if for all $x \in M$ there is a $\xi \in G$ with $\xi_* V_0 = TM_x$.*

4.3 Remark. Clearly M is a type V_0 for some V_0 if and only if G is transitive on the set of tangent spaces to M.

4.4 Recall that if M is a submanifold of the Riemannian manifold S then the second fundamental form h^M of M in S is defined as follows; let ∇^S be the Riemannian connection on S and ∇^M the Riemannian connection of M then for smooth vector fields X, Y on S

$$\nabla^S_X Y = \nabla^M_X Y + h(X, Y) \tag{4-1}$$

where $\nabla^M_X Y$ is the tangent and $h(X, Y)$ the normal component to TM of $\nabla^S_X Y$. For each $x \in M$ h_x is a symmetric bilinear from $TM_x \times TM_x$ to $T^\perp M_x$.

4.5 Let V_0 be a linear subspace of $T(G/K)_{\mathbf{o}}$ and define $\mathrm{II}(V_0)$ to be

$$\mathrm{II}(V_0) = \text{vector space of all symmetric bilinear forms from } V_0 \times V_0 \text{ to } V_0^\perp.$$

The elements of $\mathrm{II}(V_0)$ can be thought of as the second fundamental forms of submanifolds of G/K which pass through $\mathbf{o}$ and have V_0 as tangent space at $\mathbf{o}$.

Let $K(V_0)$ be the subgroup of K of elements that stabilize V_0, that is

$$K(V_0) = \{a \in K : a_* V_0 = V_0\} \tag{4-2}$$

The group $K(V_0)$ acts on $\mathrm{II}(V_0)$ in the natural way, that is for $a \in K(V_0)$ and $h \in \mathrm{II}(V_0)$ then ah is given by

$$(ah)(u, v) = a_* h(a_*^{-1} u, a_*^{-1} v). \tag{4-3}$$

Since $\mathrm{II}(V_0)$ is a vector space it makes sense to speak of polynomials on $\mathrm{II}(V_0)$. Then a polynomial $\mathcal{P}$ is **invariant** under $K(V_0)$ if and only if $\mathcal{P}(ah) = \mathcal{P}(h)$ for all $a \in K(V_0)$.

4.5 Let V_0 be a p dimensional subspace of $T(G/K)$ and let M be a submanifold of G/K of type V_0. Then for each $x \in M$ there is a $\xi \in G$ with $\xi_* V_0 = TM_x$. Thus $\xi^{-1} M$ is a submanifold of G/K through $\mathbf{o}$ whose tangent space at $\mathbf{o}$ is V_0. Therefore $h_{\mathbf{o}}^{\xi^{-1}M} \in \mathrm{II}(V_0)$. If ξ_1 is another element of G with $\xi_{1*} V_0 = TM_x$ then $\xi_1 = \xi a$ for some $a \in K(V_0)$ and $h_{\mathbf{o}}^{\xi_1^{-1}M} = a^{-1} h_0^{\xi^{-1}M}$. Therefore if $\mathcal{P}$ is any polynomial on $\mathrm{II}(V_0)$ invariant under $K(V_0)$,

$$\mathcal{P}(h_{\mathbf{o}}^{\xi_1^{-1}M}) = \mathcal{P}(a^{-1} h_{\mathbf{o}}^{\xi^{-1}M}) = \mathcal{P}(h_{\mathbf{o}}^{\xi^{-1}M}).$$

So if $x \in M$ define $\mathcal{P}(h_x^M)$ by

$$\mathcal{P}(h_x^M) = \mathcal{P}(h_{\mathbf{o}}^{\xi^{-1}M})$$

where ξ is any element of G with $\xi_* V_0 = TM_x$. We have just shown that this is independent of the choice of ξ with $\xi_* V_0 = TM_x$. It is easy to check that if $g \in G$ then

$$\mathcal{P}(h_{gx}^{gM}) = \mathcal{P}(h_x^M). \tag{4-4}$$

The integral invariants we are interested in can now be defined.

4.6 Definition. *Let V_0 be a subspace of $T(G/K)$ and $\mathcal{P}$ a polynomial on* $\mathrm{II}(V_0)$ *which is invariant under $K(V_0)$. Then for each compact submanifold M (possibly with boundary) of G/K of type V_0 define*

$$I^{\mathcal{P}}(M) = \int_M \mathcal{P}(h^M_x)\,\Omega_M(x) \tag{4-5}$$

4.7 Remarks. (1) First note that if $g \in G$ then (4-4) implies

$$I^{\mathcal{P}}(gM) = I^{\mathcal{P}}(M). \tag{4-6}$$

Thus $I^{\mathcal{P}}(M)$ is independent of the position of M in G/K up to G motions.

(2) In the case that G/K a space of constant sectional curvature of n dimensions then many of the integral invariants that are usually encountered are of the form $I^{\mathcal{P}}$. For example, if $I^{\mathcal{P}} \equiv 1$ then $I^{\mathcal{P}}(M) = \mathrm{Vol}(M)$. Also the integral invariants that appear in the Weyl tube formula and the integral of the square of the length of the second fundamental form or of the mean curvature vector are of this form.

To define these integral invariants for submanifolds M of G/K even when G is not transitive on the tangent spaces to M we extend the second fundamental form of M at x to a bilinear map of $T(G/K)_x \times T(G/K)_x$ with values in $T(G/K)_x$.

4.8 Definition. *If M is a submanifold of some Riemannian manifold S then the* **extended second fundamental form** H^M_x *of M in S at x is the symmetric bilinear form from $TS_x \times TS_x$ to TS_x given by*

$$H^M_x(u,v) = h^M_x(Pu, Pv)$$

where P is the orthogonal projection of TS_x onto TM_x.

4.9 With this definition the extension of our definitions is easy. Let

$\mathrm{EII}(T(G/K)_{\mathbf{o}})$ =vector space of symmetric bilinear forms from $T(G/K)_{\mathbf{o}} \times T(G/K)_{\mathbf{o}}$ to $T(G/K)_{\mathbf{o}}$

Then K acts on $\mathrm{EII}(T(G/K)_{\mathbf{o}})$ in the same way that $K(V_0)$ acted on $\mathrm{II}(V_0)$. If M is a submanifold of G/K, $x \in M$ and $\xi \in G$ with $\xi(\mathbf{o}) = x$ then $H^{\xi^{-1}M}_{\mathbf{o}} \in \mathrm{EII}(T(G/K)_{\mathbf{o}})$. Moreover, if $\mathcal{P}$ is a polynomial on $\mathrm{EII}(T(G/K)_{\mathbf{o}})$ which is invariant under K then $\mathcal{P}(H^M_x)$ can be defined by

$$\mathcal{P}(H^M_x) = \mathcal{P}(H^{\xi^{-1}M}_{\mathbf{o}})$$

where ξ is any element of G with $\xi(\mathbf{o}) = x$ and this definition will be independent of the choice of x. Therefore $I^{\mathcal{P}}(M)$ can be defined just as before by

$$I^{\mathcal{P}}(M) = \int_M \mathcal{P}(H^M_x)\,\Omega_M(x).$$

For this definition we still have that $I^{\mathcal{P}}(gM) = I^{\mathcal{P}}(M)$ for all $g \in G$.

4.10 The Kinematic Formula. *Let G, K, G/K be as in paragraph 3.1 and also assume that G is unimodular. Let V_0, W_0 be linear subspaces of $T(G/K)_{\mathbf{o}}$ with* $\dim(V_0) + \dim(W_0) \geq \dim(G/K)$ *and $\mathcal{P}$ be a polynomial on* $\mathrm{EII}(T(G/K)_{\mathbf{o}})$ *such that*

(a) *$\mathcal{P}$ is homogeneous of degree l, invariant under K and,*
(b)
$$\int_K \sigma(V_0^\perp, a_* W_0^\perp)^{1-l}\, \Omega_K(a) < \infty$$

Then there is a finite set of pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ such that:

(1) *each $\mathcal{Q}_\alpha$ is a homogeneous polynomial on* $\mathrm{II}(V_0)$ *invariant under $K(V_0)$,*
(2) *each $\mathcal{R}_\alpha$ is a homogeneous polynomial on* $\mathrm{II}(W_0)$ *invariant under $K(W_0)$,*
(3)
$$\mathrm{degree}(\mathcal{Q}_\alpha) + \mathrm{degree}(\mathcal{R}_\alpha) = l$$
for each α, and
(4) *for all compact submanifolds M of G/K of type V_0 and N of type W_0 (they may have boundaries) the kinematic formula*

$$\int_G I^{\mathcal{P}}(M \cap gN)\, \Omega_G = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N) \tag{4-7}$$

holds.

4.11 Remark. If all the polynomials on $\mathrm{II}(V_0)$ which are invariant under $K(V_0)$ and all the polynomials on $\mathrm{II}(W_0)$ invariant under $K(W_0)$ are known then for a given polynomial $\mathcal{P}$ on $\mathrm{EII}(T(G/K)_{\mathbf{o}})$ invariant under K and homogeneous of degree l it is in theory possible to prove a kinematic formula for $\int_G I^{\mathcal{P}}(M \cap gN)\, \Omega_G(g)$ as follows; For each i with $0 \leq i \leq l$ let $\{\mathcal{Q}^i_\alpha\}$ be a basis for the polynomials on $\mathrm{II}(V_0)$ invariant under $K(V_0)\}$ and homogeneous of degree i and $\{\mathcal{R}^{l-i}_\beta$ a basis for the polynomials on $\mathrm{II}(W_0)$ invariant under $K(W_0)$ and homogeneous of degree $l - i$. Then by the theorem there are constants $c_{i,\alpha,\beta}$ with

$$\int_G I^{\mathcal{P}}(M \cap gN)\, \Omega_G(g) = \sum_{i,\alpha,\beta} c_{i,\alpha,\beta} I^{\mathcal{Q}^i_\alpha}(M) I^{\mathcal{R}^{l-i}_\beta}(N) \tag{4-8}$$

for all compact submanifolds M, N with M of type V_0 and N of type W_0. By then evaluating both sides of this equation for several choices of submanifolds M, N it is possible to get enough equations to solve for the $c_{i,\alpha,\beta}$'s. This last step is clearly formidable and is to be avoided if possible. Alternately theorem 7.2 below can be used to evaluate $\int_G I^{\mathcal{P}}(M \cap gN)\, \Omega_G(g)$. In practice it seems that a combination of these two methods works the best. First use theorem 7.2 and the form of the particular polynomial $\mathcal{P}$ to conclude that most of the $c_{i,\alpha,\beta}$ are zero. Then evaluate both sides of (4-8) for M and N having enough symmetry that the calculations are manageable. As a nontrivial example of this we will use these methods to give a new proof of the kinematic formula of Chern and Federer that works in all spaces of constant sectional curvature and not only in Euclidean space.

4.12 The Transfer Principle. The set up here is similar to that of paragraph 3.5. That is G' is another unimodular Lie group of the same dimension as G and K' is a compact subgroup of G' the same dimension as K. Assume there is a smooth isomorphism $\rho : K \to K'$ and a linear isometry $\psi : T(G/K)_{\mathbf{o}} \to T(G'/K')_{\mathbf{o}'}$ such that

(a) $\mathrm{Vol}(K) = \mathrm{Vol}(K')$ (no other assumption is put on the metrics on K and K'),
(b) $\psi \circ a_* = \rho(a)_* \circ \psi$ for all $a \in K$.

Let V_0 and W_0 be linear subspaces of $T(G/K)_{\mathbf{o}}$ and set $V_0' = \psi V_0$ and $W_0' = \psi W_0$. The map ψ induces isomorphisms of $\mathrm{II}(V_0)$, $\mathrm{II}(W_0)$ and $\mathrm{EII}(T(G/K)_{\mathbf{o}})$ onto $\mathrm{II}(V_0')$, $\mathrm{II}(W_0')$ and $\mathrm{EII}(T(G'/K')_{\mathbf{o}'})$ respectively and thus isomorphisms of the rings of polynomials on $\mathrm{II}(V_0)$, $\mathrm{II}(W_0)$ and $\mathrm{EII}(T(G/K)_{\mathbf{o}})$ onto the rings of polynomials on $\mathrm{II}(V_0')$, $\mathrm{II}(W_0')$ and $\mathrm{EII}(T(G'/K')_{\mathbf{o}'})$ respectively. If $\mathcal{P}$ is a polynomial on $\mathrm{II}(V_0)$ then let $\mathcal{P}'$ be the polynomial on $\mathrm{II}(V_0')$ which is the image of $\mathcal{P}$ under this isomorphism. Likewise for polynomials on $\mathrm{II}(W_0)$ and $\mathrm{EII}(T(G/K)_{\mathbf{o}})$. See paragraph 6.8 below for the details involving this isomorphism.

Condition (b) above implies $\rho K[V_0] = K'[V_0']$, $\rho K[W_0] = K'[W_0']$ and if $\mathcal{P}$ is a polynomial on $\mathrm{II}(V_0)$ (resp. $\mathrm{II}(W_0)$ or $\mathrm{EII}(T(G/K)_{\mathbf{o}})$) then $\mathcal{P}$ is invariant under $K[V_0]$ (resp. $K[W_0]$ or K) if and only if $\mathcal{P}'$ is invariant under $K'[V_0']$ (resp. $K'[W_0']$ or K').

With this notation the duality principle can be stated. Assume the formula (4-7) holds in G/K then for very compact submanifold M' of G'/K' of type V_0' and compact submanifold N' of G'/K' of type W_0' (they may have boundaries) the kinematic formula

$$\int_{G'} I^{\mathcal{P}'}(M' \cap gN')\, \Omega_{G'}(g) = \sum_{\alpha} I^{\mathcal{Q}'_\alpha}(M') I^{\mathcal{R}'_\alpha}(N') \tag{4-9}$$

holds.

Appendix to Section 4: Crofton Type Kinematic Formulas.

4.13 In this section we will use the same notation as in the appendix to section 3 except that we make the additional assumption that G is unimodular. Using the methods of that appendix we sketch a proof of the following:

4.14 Crofton and Linear Kinematic Formula. *Let V_0 be a subspace of $T(G/K)_{\mathbf{o}}$ such that $\dim(V_0) + \dim(L_0) \geq \dim(G/K)$. Let $\mathcal{P}$ be a polynomial on $\mathrm{EII}(T(G/K))_{\mathbf{o}}$ which is*

(a) *homogeneous of degree l and invariant under K and assume*
(b) $\int_K \sigma(V_0^\perp, a_*^{-1} W_0^\perp)^{1-l}\, \Omega_K(a) < \infty$ *(where $W_0 = T(L_0)$).*

Then there are polynomials $\mathcal{Q}_0, \dots, \mathcal{Q}_l$ on $\mathrm{II}(V_0)$ such that

(1) *$\mathcal{Q}_i$ is homogeneous of degree i and is invariant under $K(V_0)$,*
(2) *for each compact submanifold M of G/K of type V_0 (possibly with boundary) the formula*

$$\int_{G/G(L_0)} I^{\mathcal{P}}(M \cap gN)\, \Omega_{G/G(L_0)}(L) = \sum_{i=0}^{l} I^{\mathcal{Q}_i}(M) \tag{4-10}$$

holds. If in addition L_0 is totally geodesic and $l \geq 1$ then $\mathcal{Q}_i = 0$ for $0 \leq i \leq l-1$ and so the last equation reduces to

$$\int_{G/G(L_0)} I^{\mathcal{P}}(M \cap gN)\, \Omega_{G/G(L_0)}(L) = I^{\mathcal{Q}_l}(M). \tag{4-11}$$

4.15 Remark That (4-10) reduces to (4-11) when L_0 is totally geodesic justifies our earlier claim that as far as the type of integral geometric formulas that arise $G/G(L_0)$ behaves very much like a Grassmann manifold. Compare with the formulas in section 8 of [6] and the linear kinematic formula in section 3 of [19].

4.16 Outline of the proof. Proving 4.14 from 4.10 follows exactly the same steps as proving equation (3-21) from equation (3-4). As before start with equation (3-19) only this time let $h(g) = I^{\mathcal{P}}(M \cap gN_0)$. Then use theorem 4.10 to conclude that

$$\int_G h(g)\, \Omega_G(g) = \sum_{\alpha} I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N_0) \tag{4-12}$$

where the pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ are given to us by 4.10. With ξ_L and as in paragraph 3.18 we can use the invariance properties of $I^{\mathcal{P}}$,

$$\begin{aligned} h(\xi_L a) \quad &= I^{\mathcal{P}}(M \cap (\xi_L a)N_0) &&= I^{\mathcal{P}}(M \cap L \cap \xi_L a N_0) \\ &= I^{\mathcal{P}}(\xi_L^{-1} M \cap \xi_L^{-1} L \cap aN_0) &&= I^{\mathcal{P}}((\xi_L^{-1} M \cap L_0) \cap aN_0). \end{aligned}$$

This implies

$$\int_{G(L_0)} h(\xi_L a)\, \Omega_{G(L_0)}(a) = \int_{G(L_0)} I^{\mathcal{P}}((\xi_L^{-1} M \cap L_0) \cap aN_0)\, \Omega_{G(L_0)}(a) \tag{4-13}$$

We need one extra piece of information.

Lemma. *If N_0 is as in paragraph 3.18 then there is a constant c_2 such that for every compact $p+q-n$ dimensional ($p = \dim(M)$, $q = \dim(L_0)$, $n = \dim(G/K)$) submanifold M_0 of $L_0 = G(L_0)/K(L_0)$ and every continuous function $f : M_0 \to \mathbb{R}$ the formula*

$$\int_{G(L_0)} \int_{M_0 \cap aN_0} f\, \Omega_{M_0}\, \Omega_{G(L_0)}(a) = c_2 \int_{M_0} f\, \Omega_{M_0} \tag{4-14}$$

holds.

This can be proven directly from the basic integral formula 2.7 or by first assuming that f is a simple function, *i.e.* one that, except for a set of measure zero, is constant on each of a finite number of open subsets of M_0 that have well behaved boundaries. Applying the result of example 3.12(d) to each of the open sets on which f is constant will yield (4-14) in the case f is simple. The general case then follows by taking limits.

This lemma with $M_0 = \xi_L^{-1}M \cap L_0$ and $f(x) = \mathcal{P}(h_x^{\xi_L^{-1}M\cap L_0})$ $(= \mathcal{P}(h^{(\xi_L^{-1}M\cap L_0)\cap aN_0})$ when x is in the interior of aN_0) implies.

$$\begin{aligned}\int_{G(L_0)} h(\xi_L a)\,\Omega_{G(L_0)}(a) &= \int_{G(L_0)}\int_{(\xi_L^{-1}M\cap L_0)\cap aN_0} \mathcal{P}(h_x^{\xi_L^{-1}M\cap L_0})\,\Omega_{\xi_L^{-1}M\cap L_0}(x)\,\Omega_{G(L_0)}(a)\\ &= c_2\int_{\xi_L^{-1}M\cap L_0}\mathcal{P}(h_x^{\xi_L^{-1}M\cap L_0})\,\Omega_{\xi_L^{-1}M\cap L_0}(x)\\ &= c_2 I^{\mathcal{P}}(\xi_L^{-1}M\cap L_0) = c_2 I^{\mathcal{P}}(M\cap L).\end{aligned}$$

Putting this and (4-12) in (3-19) yields

$$\begin{aligned}\int_{G/G(L_0)} I^{\mathcal{P}}(M\cap L)\,\Omega_{G/G(L_0)}(L) &= \frac{c_1}{c_2}\sum_\alpha I^{\mathcal{Q}_\alpha}(M)I^{\mathcal{R}_\alpha}(N_0)\\ &= \sum_{i=0}^{l}\sum_{\deg \mathcal{Q}_\alpha = i}\frac{c_1}{c_2}I^{\mathcal{R}_\alpha}(N_0)I^{\mathcal{Q}_\alpha}(M)\end{aligned}$$

which easily implies equation (4-10). If L_0 (and thus N_0) is totally geodesic then $h_x^{N_0} = 0$ for all x. But then $\deg(\mathcal{R}_\alpha) > 0$ implies $I^{\mathcal{R}_\alpha}(N_0) = 0$. Using this in the last equation implies 4-11. This completes the proof.

5. The Second Fundamental Form of an Intersection

5.1 The first task toward proving the kinematic formula is to get an explicit formula for the second fundamental form of a transverse intersection $M\cap N$ in terms of the second fundamental forms of M and N. An estimate on the length of the second fundamental form of $M\cap N$ in terms of the second fundamental forms of M and N and the angle $\sigma(T^\perp M_x, T^\perp N_x)$ will also be needed.

5.2 If S is a smooth Riemannian manifold and M a smooth submanifold of S then recall that the length of the second fundamental form h_x^M of M at x is defined by

$$\|h_x^M\|^2 = \sum_{1\le i,j\le p}\|h_x^M(e_i,e_j)\|^2$$

where $p = \dim(M)$, $n = \dim(S)$ and $e_1,\dots,e_p$ is an orthonormal basis of TM_x. Recalling definition 4.8 of the extended second fundamental form H_x^M of M at x we define its length to be

$$\|H_x^M\|^2 = \sum_{1\le i,j\le n}\|H_x^M(e_i,e_j)\|^2$$

where $e_1,\dots,e_n$ is any orthonormal basis of TS_x. It is left to the reader to verify that $\|H_x^M\| = \|h_x^M\|$.

5.3 Definition. *Let V and W be linear subspaces of a finite dimensional real inner product space T such that $V + W = T$. Then define P_W^V by*

(5-1) $$P_W^V = \text{projection onto } (V\cap W)^\perp\cap W \text{ with kernel } V$$

Note that $(V\cap W)^\perp = V^\perp + W^\perp$ and that there is a direct sum decomposition

(5-2) $$T = V\oplus((V^\perp\oplus W^\perp)\cap W) = V\oplus((V\cap W)^\perp\cap W)$$

and therefore this definition makes sense.

5.4 Proposition. *Let S be a smooth Riemannian manifold and M and N submanifolds of S that have nonempty transverse intersection. Then for each $x \in M \cap N$ the second fundamental form of $M \cap N$ is given by*

$$h^{M\cap N}(X,Y) = P^{TM_x}_{TN_x} h^M_x(X,Y) + P^{TN_x}_{TM_x} h^N_x(X,Y) \tag{5-3}$$

for all $X, Y \in T(M\cap N)_x = TM_x \cap TN_x$ and therefore the extended second fundamental form of $M \cap N$ is given by

$$H^{M\cap N}_x(X,Y) = P^{TM}_{TN} h^M_x(PX,PY) + P^{TN}_{TM} h^N_x(PX,PY)$$

where $P : TS_x \to T(M\cap N)_x$ is orthogonal projection. Also

$$\|h^{M\cap N}_x\| = \|H^{M\cap N}_x\| \le \frac{\sqrt{2}}{\sigma(T^\perp M_x, T^\perp N_x)} (\|h^M_x\|^2 + \|h^N_x\|^2)^{\frac{1}{2}} \tag{5-4}$$

5.5 Remark In the case that S is a Euclidean space the equation (5-3) is equivalent to formula (3) on page 112 of Chern's paper [6], however the notation is much different.

5.6 Proof. Let ∇^S, ∇^M, ∇^N, $\nabla^{M\cap N}$ be the Riemannian connections of the indicated manifolds. Then for smooth vector fields X, Y on $M \cap N$ defined near x

$$\begin{aligned} \nabla^S_X Y &= \nabla^M_X Y + h^M(X,Y) \\ \nabla^S_X Y &= \nabla^N_X Y + h^N(X,Y) \\ \nabla^S_X Y &= \nabla^{M\cap N}_X Y + h^{M\cap N}(X,Y) \end{aligned} \tag{5-5}$$

The vector $H^{M\cap N}(X,Y)$ is in $T^\perp(M\cap N) = (TM\cap TN)^\perp = ((TM\cap TN)^\perp \cap TM) \oplus ((TM\cap TN)^\perp \cap TN)$. Therefore $h^{M\cap N}(X,Y)$ can be decomposed as $h^{M\cap N}(X,Y) = Z_1 + Z_2$ with $Z_1 \in (TM\cap TN)^\perp \cap TM$ and $Z_2 \in (TM\cap TN)^\perp \cap TN$. Whence

$$\begin{aligned} &P^{TM}_{TN}\nabla^{M\cap N}_X Y = 0, \quad P^{TM}_{TN} Z_1 = 0, \qquad P^{TM}_{TN} Z_2 = Z_2 \\ &P^{TN}_{TM}\nabla^{M\cap N}_X Y = 0 \quad P^{TN}_{TM} Z_1 = Z_1, \qquad P^{TM}_{TN} Z_2 = 0 \\ &P^{TM}_{TN}\nabla^M_X Y = 0, \quad P^{TM}_{TN}\nabla^N_X Y = 0 \end{aligned} \tag{5-6}$$

Using (5-5) and (5-6)

$$\begin{aligned} h^{M\cap N}(X,X) &= Z_1 + Z_2 \\ &= P^{TM}_{TN}(\nabla^{M\cap N}_X Y + Z_1 + Z_2) + P^{TN}_{TM}(\nabla^{M\cap N}_X Y + Z_1 + Z_2) \\ &= P^{TM}_{TN}(\nabla^S_X Y) + P^{TN}_{TM}(\nabla^S_X Y) \\ &= P^{TM}_{TN}(\nabla^M_X Y + h^M(X,Y)) + P^{TN}_{TM}(\nabla^N_X Y + h^N(X,Y)) \\ &= P^{TM}_{TN} h^M(X,Y) + P^{TN}_{TM} h^N(X,Y) \end{aligned}$$

which completes the proof of equation (5-3).

The inequality (5-4) requires more work. We start with

5.7 Lemma. *With the notation of definition 5.3*

$$\|P_V^W X\| \le \frac{1}{\sigma(V^\perp, W^\perp)}\|X\|$$

for all $X \in T$.

Proof. First note $T = (V \cap W) \oplus (V \cap W)^\perp$, $P_W^V(V \cap W) = \{0\}$, and that $(V \cap W)^\perp$ is stable under P_W^V. Thus in proving the lemma T can be replaced by $(V \cap W)^\perp$, V by $V \cap (V \cap W)^\perp$ and W by $W \cap (V \cap W)^\perp$. Then $T = V \oplus W$ and P_W^V is the projection of T onto W with kernel V. Also in this case $T = V^\perp \oplus W^\perp$. Let $p = \dim(V) = \dim(W^\perp)$ and $q = \dim(W) = \dim(V^\perp)$. Then we claim that it is possible to choose an orthonormal basis $v_1, \ldots, v_q$ of $V^\perp$ and an orthonormal basis $w_1, \ldots, w_q$ of W in such a way that

$$\langle v_i, w_j \rangle = 0 \qquad i \ne j, \qquad 1 \le i, j \le q. \tag{5-8}$$

To see this start with arbitrary orthonormal bases $v_1', \ldots, v_q'$ of $V^\perp$ and $w_1', \ldots, w_q'$ of W. If $P = [p_{ij}]$ and $Q = [q_{ij}]$ are any $q \times q$ orthogonal matrices, $v_i = \sum_j p_{ij} v_j'$, $w_i = \sum_J q_{ij} w_j$, A' is the matrix with entries $a_{ij} = \langle v_i, v_j \rangle$, and A is the matrix with entries $a_{ij} = \langle v_i, v_j \rangle$ then $v_1, \ldots, v_q$ is an orthonormal basis of $V^\perp$, $w_1, \ldots, w_q$ is an orthonormal basis of W and a little calculation shows $A = PA'Q^t$ where Q^t is the transpose of W. It is well known that any matrix A' can be factored as $A' = HU$ with H symmetric and U orthogonal and that any symmetric matrix H can be written as $H = U_1 D U_1^t$ where D is diagonal and U_1 orthogonal. If we set $P = U_1^t$ and $Q = U_1^t U$ then P and Q are orthogonal and $A = PA'Q^t = U_1^t(U_1 D U_1^t U)(U_1^t U)^t = U_1^t U_1 D U_1^t U U^t U_1 = D$. But A being a diagonal matrix is easily seen to be equivalent to the orthogonality relationships (5-8).

Complete $v_1, \ldots, v_q$ to $v_1, \ldots, v_{p+q}$ and $w_1, \ldots, w_q$ to $w_1, \ldots, w_{p+q}$ orthonormal bases of T. Then $v_{q+1}, \ldots, v_{p+q}$ is an orthonormal basis of V and $w_{q+1}, \ldots, w_{p+q}$ is an orthonormal basis of $W^\perp$. From the definition of P_W^V it follows $P_W^V v_j = 0$ for $q+1 \le j \le p+q$ and $P_W^V w_i = w_i$ for $1 \le i \le q$. Using these facts and the orthogonality relations (5-8) it follows for $1 \le i \le q$ that

$$\begin{aligned} w_i = P_W^V w_i &= P_W^V\left(\sum_{j=1}^{q} \langle w_i, v_j \rangle v_j\right) \\ &= P_W^V\left(\langle w_i, v_i \rangle v_i + \sum_{j=q+1}^{p+q} \langle w_i, v_j \rangle v_j\right) \\ &= \langle w_i, v_i \rangle P_W^V v_i + 0 \end{aligned}$$

and thus

$$\begin{aligned} P_W^V v_i &= \frac{1}{\langle w_i, v_i \rangle} w_i, \quad && 1 \le i \le q \\ P_W^V v_i &= 0, && q+1 \le i \le p+q \end{aligned}$$

Therefore if $x \in T$ is written as $x = \sum_{i=1}^{p+q} x_i v_i$ then these last equations imply (as $v_1, \dots, v_{p+q}$ and $w_1, \dots, W_{p+q}$ are both orthonormal) that $\|x\| = \sum_{i=1}^{p+q} (x_i)^2$ and thus

$$\begin{aligned}
\|P_W^V x\| &= \sum_{j=1}^{q} \frac{(x_j)^2}{\langle w_j, v_j \rangle^2} \\
&\le \max_{1 \le i \le q} \frac{1}{\langle w_i, v_i \rangle^2} \sum_{j=1}^{p+q} (x_j)^2 \\
&= \max_{1 \le i \le q} \frac{1}{\langle w_i, v_i \rangle^2} \|x\|^2
\end{aligned}$$

Relabel so that $|\langle w_1, v_1 \rangle|$ is the smallest of $|\langle w_1, v_1 \rangle|, \dots, |\langle w_q, v_q \rangle|$ then we have just shown that

$$\|P_W^V x\| \le \frac{1}{|\langle w_1, v_1 \rangle|} \|x\| \tag{5-9}$$

The vectors $v_1, \dots, v_q$ are an orthonormal basis of $V^\perp$ and $w_{q+1}, \dots, w_{p+q}$ is an orthonormal basis of $W^\perp$. The relations (5-8) yield that for $1 \le i \le q$

$$v_i = \langle v_i, w_i \rangle w_i + \sum_{j=q+1}^{p+q} w_i = \langle v_i, w_i \rangle w_i + \widehat{w}_i$$

where $\widehat{w}_i$ is in the span of $w_{q+1}, \dots, w_{p+q} = 0$ and thus $\widehat{w}_i \wedge w_{q+1} \wedge \dots \wedge w_{p+q}$. Whence

$$\begin{aligned}
\sigma(V^\perp, W^\perp) &= \|v_1 \wedge \dots \wedge v_q \wedge w_{q+1} \wedge \dots \wedge w_{p+q}\| \\
&= \|(\langle v_1, w_1 \rangle w_1 + \widehat{w}_1) \wedge \dots \wedge (\langle v_q, w_q \rangle w_q + \widehat{w}_q) \wedge w_{q+1} \wedge \dots \wedge_{p+q} \| \\
&= |\langle v_1, w_1 \rangle| \cdots |\langle v_q, w_q \rangle| \|w_1 \wedge \dots \wedge w_{p+q}\| \\
&= |\langle v_1, w_1 \rangle| \cdots |\langle v_q, w_q \rangle|
\end{aligned}$$

Therefore

$$\frac{1}{|\langle v_1, w_1 \rangle|} = \frac{|\langle v_2, w_2 \rangle| \cdots |\langle v_q, w_q \rangle|}{\sigma(V^\perp, W^\perp)} \le \frac{1}{\sigma(V^\perp, W^\perp)}$$

as each $|\langle v_i, w_i \rangle| \le 1$ by the Cauchy-Schwartz inequality. Using this inequality in (5-9) completes the proof of the lemma.

5.8 We now prove the inequality (5-4). Let $k = \dim(T(M \cap N)_x)$, $P : TS_x \to T(M \cap N)_x$ be the orthogonal projection and $e_1, \dots, e_n$ ($n = \dim(S)$) be an orthonormal basis of TS_x such that $e_1, \dots, e_k$ is an orthonormal basis of $T(M \cap N)_x$. Then the following two inequalities are elementary

$$\begin{aligned}
&\sum_{1 \le i,j \le n} \|h_x^M(Pe_i, Pe_j)\|^2 = \sum_{1 \le i,j \le k} \|h_x^M(e_i, e_j)\|^2 \le \|h_x^M\|^2 \\
&\sum_{1 \le i,j \le n} \|h_x^N(Pe_i, Pe_j)\|^2 = \sum_{1 \le i,j \le k} \|h_x^N(e_i, e_j)\|^2 \le \|h_x^N\|^2 \qquad (5\text{-}10)
\end{aligned}$$

Now use the last lemma, the form of $h^{M\cap N}$ given by (5-3) and the elementary inequality $(x+y)^2 \le 2(x^2+y^2)$

$$\begin{aligned}
\|h_x^{M\cap N}\|^2 = \|H_x^{M\cap N}\|^2 &= \sum_{i,j} \|P_{TN}^{TM} h^M(Pe_i, Pe_j) + P_{TM}^{TN} h^N(Pe_i, Pe_j)\|^2 \\
&\le \sum_{i,j} (\|P_{TN}^{TM} h^M(Pe_i, Pe_j)\| + \|P_{TM}^{TN} h^N(Pe_i, Pe_j)\|)^2 \\
&\le 2\sum_{i,j} (\|P_{TN}^{TM} h^M(Pe_i, Pe_j)\|^2 + \|P_{TM}^{TN} h^N(Pe_i, Pe_j)\|^2) \\
&\le \frac{2}{\sigma(T^\perp M_x, T^\perp N_x)^2} \sum_{i,j} (\|h^M(Pe_i, Pe_j)\|^2 + \|h^N(Pe_i, Pe_j)\|^2) \\
&\le \frac{2}{\sigma(T^\perp M_x, T^\perp N_x)^2} (\|h^M\|^2 + \|h^N\|^2)
\end{aligned}$$

where the last line use the inequalities (5-10). This completes the proof of proposition 5.4.

6. Lemmas and Definitions

6.1 In this section we establish the notation and prove the lemmas that will be needed to prove the kinematic formula and the transfer principle. For the rest of this section the following notation will be used:

$$T = n \text{ dimensional real inner product space.}$$

Then, as in section 4, for any subspace V_0 of T set

$$\mathrm{II}(V_0) = \text{vector space of symmetric bilinear forms } V_0 \times V_0 \to V_0^\perp$$

and

$$\mathrm{EII}(T) = \text{symmetric bilinear forms } T \times T \to T.$$

For $0 \le p \le n$ set

$$\begin{aligned}\mathrm{II}_p(T) =&\text{set of pairs } (V, h) \text{ where } V \text{ is a p dimensional subspace of } T \text{ and } h \text{ is} \\ &\text{a symmetric bilinear form } V \times V \to V^\perp\end{aligned}$$

If $(V, h) \in \mathrm{II}_p(T)$ and W is any linear subspace of T such that $V + W = T$ then define $G_W(V, h)$, the **geodesic section of** (V, h) **by** W, to be the element of $\mathrm{EII}(T)$ given by

$$G_W(V, h)(u, v) = P_W^V h(Pu, Pv)$$

where P_W^V is defined by equation (5-1) and P is the orthogonal projection of T onto $V \cap W$.

In the case that T is the tangent space to some n dimensional Riemannian manifold S at a point x then the above objects have geometric meaning. The vector space $\mathrm{II}(V_0)$ can be thought of as the set of all second fundamental forms of all submanifolds M passing through x with $TM_x = V_0$. The set $\mathrm{II}_p(T)$ can be viewed as the set of pairs (TM_x, h_x^M) for all p dimensional submanifolds M passing through x.

If S has constant sectional curvature, $(V,h) = (TM_x, h_x^M) \in \mathrm{II}_p(TS_x)$ for some p dimensional submanifold M of S, and W is a linear subspace of TS_x with $W + TM_x = TS_x$ then there is a totally geodesic submanifold N of S through x with $TN_x = W$. By proposition 5.4 the extended second fundamental form of $M \cap N$ at x is $G_{TN_x}(TM_x, h_x^M) = G_W(V,h)$, and therefore $G_W(V,h)$ is the extended second fundamental form of "a totally geodesic section of M in the direction W".

Also if M and N are submanifolds of S intersecting transversely at x then by proposition 5.4 and the notation just introduced the extended second fundamental form of $M \cap N$ at x is

$$\begin{aligned} H_x^{M\cap N} &= G_{TN_x}(TM_x, h_x^M) + G_{TM_x}(TN_x, h_x^N) \\ &= G_W(V,h_1) + G_V(W,h_2) \end{aligned}$$

where $(TM_x, h_x^M) = (V, h_1)$ and $(TN_x, h_x^N) = (W, h_2)$.

6.2 Fix a compact Lie group K with volume form Ω_K and let $a \to a_*$ be an orthogonal representation of K on T. Then K acts on $\mathrm{II}_p(T)$ by

$$a(V,h) = (a_*V, ah)$$

where $a \in K$, $(V,h) \in \mathrm{II}_p(T)$ and ah is given by

$$(ah)(u,v) = a_*h(a_*^{-1}u, a_*^{-1}v).$$

Also K acts on $\mathrm{EII}(T)$ by letting aH (for $a \in K$, $H \in \mathrm{EII}(T)$) be defined by the last equation with h replaced by H. If $(V,h) \in \mathrm{II}_p(T)$ and W is a subspace of T with $V + W = T$ then a chase through the definitions shows that for all $a \in K$

$$a(G_W(V,h)) = G_{a_*W}(a_*V, ah). \tag{6-2}$$

6.3 Definition. *If $p+q \geq n$, $(V,h_1) \in \mathrm{II}_p(T)$, $(W,h_2) \in \mathrm{II}_q(T)$ and $\mathcal{P}$ is a polynomial on* $\mathrm{EII}(T)$ *that is invariant under K then define*

(6-2)

$$I_K^{\mathcal{P}}(V,h_1,W,h_2) = \int_K \mathcal{P}(G_{b_*^{-1}W}(V,h_1) + G_V(b_*^{-1}W, b^{-1}h_2))\sigma(V^\perp, b_*^{-1}W^\perp)\,\Omega_K(b)$$

provided this integral converges.

6.4 Lemma. *Let $(V,h_1) \in \mathrm{II}_p(T)$ and $(W,h_2) \in \mathrm{II}_q(T)$ and assume that*

$$\int_K \sigma(V^\perp, b_*^{-1}W^\perp)^{1-l}\,\Omega_K(b) < \infty. \tag{6-4}$$

Then for every polynomial $\mathcal{P}$ on $\mathrm{EII}(T)$ *which is both homogeneous of degree l and invariant under K, the integral defining $I_K^{\mathcal{P}}(V,h_1,W,h_2)$ converges and*

(1) $I_K^{\mathcal{P}}(V,h_1,W,h_2) = I_K^{\mathcal{P}}(W,h_2,V,h_1)$,

(2) *for all* $a \in K$

$$I_K^{\mathcal{P}}(a_*V, ah_1, W, h_2) = I_K^{\mathcal{P}}(V, h_1, a_*W, ah_2) = I_K^{\mathcal{P}}(V, h_1, W, h_2),$$

(3) *for* $a \in K$

$$\begin{aligned} a_*V = V \qquad &\text{implies} \qquad I_K^{\mathcal{P}}(V, ah_1, W, h_2) = I_K^{\mathcal{P}}(V, h_1, W, h_2) \\ a_*W = W \qquad &\text{implies} \qquad I_K^{\mathcal{P}}(V, h_1, W, ah_2) = I_K^{\mathcal{P}}(V, h_1, W, h_2) \end{aligned}$$

Proof. Translating the bound of proposition 4.5 on the length of the second fundamental form of an intersection into the present context (see equation (6-1)) for all $b \in K$ with $\sigma(V, b_*^{-1}W) \neq 0$ it follows that

$$\|G_{g_*^{-1}W}(V, h_1) + G_V(b_*^{-1}W, bh_2)\| \leq \frac{\sqrt{2}}{\sigma(V^\perp, b_*^{-1}W^\perp)}(\|h_1\|^2 + \|h_2\|^2)^{\frac{1}{2}}.$$

Because $\mathcal{P}$ is homogeneous of degree l there is a constant $C(\mathcal{P})$, only depending on $\mathcal{P}$, such that

(6-5)

$$|\mathcal{P}(G_{b_*^{-1}W}(V, h_1) + G_V(b_*^{-1}W, b^{-1}h_2))| \leq C(\mathcal{P})(\|h_1\|^2 + \|h_2\|^2)^{\frac{l}{2}}\sigma(V^\perp, b_*^{-1}W^\perp)^{-1}$$

Use this inequality in equation (6-3) to conclude that the integral defining $I_K^{\mathcal{P}}(V, h_1, W, h_2)$ converges.

To prove (1) compute;

$$\begin{aligned} &I_K^{\mathcal{P}}(W, h_2, V, h_1) \\ &\quad = \int_K \mathcal{P}(G_{b_*^{-1}V}(W, h_2) + G_W(b_*^{-1}V, b^{-1}h_1))\sigma(W^\perp, b_*^{-1}V^\perp)\,\Omega_K(b) \\ &\quad = \int_K \mathcal{P}(b[G_{b_*^{-1}V}(W, h_2) + G_W(b_*^{-1}V, b^{-1}h_1)])\sigma(W^\perp, b_*^{-1}V^\perp)\,\Omega_K(b) \\ &\quad = \int_K \mathcal{P}(G_{b_*W}(V, h_1) + G_V(b_*W, bh_2))\sigma(V^\perp, b_*W^\perp)\,\Omega_K(b) \\ &\quad = \int_K \mathcal{P}(G_{b_*^{-1}W}(V, h_1) + G_V(b_*^{-1}W, b^{-1}h_2))\sigma(W^\perp, b_*^{-1}V^\perp)\,\Omega_K(b) \\ &\quad = I_K^{\mathcal{P}}(V, h_1, W, h_2) \end{aligned}$$

where going from the second line to the third used the invariance of $\mathcal{P}$ under K, going from the third line to the fourth used the equation (6-2) and that $\sigma(W^\perp, b_*^{-1}V^\perp) = \sigma(V^\perp, b_*W^\perp)$ (Which follows from equation (2-2) with $\rho = b_*$), and going from the fourth to the fifth line is just the change of variable $b \mapsto b^{-1}$ (K is compact and thus unimodular).

To prove (2)

$$\begin{aligned} &I_K^{\mathcal{P}}(W, h_1, a_*W, ah_2) \\ &\quad = \int_K \mathcal{P}(G_{b_*^{-1}a_*W}(V, h_1) + G_V(b_*^{-1}a_*W, b^{-1}ah_2))\sigma(V^\perp, b_*^{-1}W^\perp)\,\Omega_K(b) \\ &\quad = \int_K \mathcal{P}(G_{b_*^{-1}W}(V, h_1) + G_V(b_*^{-1}W, b^{-1}h_2))\sigma(V^\perp, b_*^{-1}W^\perp)\,\Omega_K(b) \\ &\quad = I_K^{\mathcal{P}}(V, h_1, W, h_2) \end{aligned}$$

where all that was needed this time was the change of variable $b \mapsto ab$. This last equation together with (1) implies (2) and (2) implies (3). This completes the proof.

Having set up all this notation we can now give the lemma that does all the rest of the work needed to prove the kinematic formula that was not done in sections 2 and 3.

6.5 LEMMA. *Let V_0 and W_0 be linear subspaces of T and $\mathcal{P}$ a polynomial on* $\mathrm{EII}(T)$ *such that*

(a) *$\mathcal{P}$ is homogeneous of degree l, invariant under K and,*
(b)

$$\int_K \sigma(V_0^\perp, b_*^{-1}W_0^\perp)^{1-l}\,\Omega_K(b) < \infty$$

Then there is a finite set of pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ such that;

(1) *each $\mathcal{Q}_\alpha$ is a homogeneous polynomial on* $\mathrm{II}(V_0)$ *invariant under* $K(V_0) = \{a \in K : a_*V_0 = V_0\}$,
(2) *each $\mathcal{R}_\alpha$ is a homogeneous polynomial on* $\mathrm{II}(W_0)$ *invariant under* $K(W_0)$,
(3) $\mathrm{degree}(\mathcal{Q}_\alpha) + \mathrm{degree}(\mathcal{R}_\alpha) = l$ *for each α and*
(4) *for all $h_1 \in \mathrm{II}(V_0)$ and $h_2 \in (W_0)$*

$$I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2) = \sum_\alpha \mathcal{Q}_\alpha(h_1)\mathcal{R}_\alpha(h_2).$$

6.6 Remark The statement of this lemma has been made to parallel the statement of 4.10 to emphasize that the form of a kinematic formula for submanifolds of a homogeneous space G/K does not depend on the full group of transformations G, but that the form of the kinematic formula is dictated by the invariant theory of the isotropy subgroup K.

6.7 PROOF. For the moment fix $b \in K$ such that $V_0 + b_*^{-1}W_0 = T$. Then the map on $\mathrm{II}(V_0) \times \mathrm{II}(W_0)$ given by

$$(h_1, h_2) \mapsto G_{b_*^{-1}W_0}(V_0, h_1) + G_{V_0}(b_*^{-1}W_0, b^{-1}h_2)$$

is a linear (it is trivial that $G_{W_0}(V_0, h_1 + h_1') = G_{W_0}(V_0, h_1) + G_{W_0}(V_0, h_1')$ etc.) from $\mathrm{II}(V_0) \times \mathrm{II}(W_0)$ to $\mathrm{EII}(T)$. Because $\mathcal{P}$ is homogeneous of degree l it follows that the map

$$(h_1, h_2) \mapsto \mathcal{P}(G_{b_*^{-1}W_0}(V_0, h_1) + G_{V_0}(b_*^{-1}W_0, b^{-1}h_1))\sigma(V_0^\perp, b_*^{-1}W_0^\perp)$$

is a polynomial, homogeneous of degree l, on $\mathrm{II}(V_0) \times \mathrm{II}(W_0)$ whose coefficients depend on b. Integration with respect to b over the group K (this integral exists by lemma 6.4) eliminates the dependence on b and the result that is the map $(h_1, h_2) \mapsto I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2)$ which must also then have the homogeneity property

$$I_K^{\mathcal{P}}(V_0, \lambda h_1, W_0, \lambda h_2) = \lambda^l I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2) \tag{6-6}$$

By choosing bases for $\mathrm{II}(V_0)$ and $\mathrm{II}(W_0)$ and writing the polynomial

$$(h_1, h_2) \mapsto I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2)$$

in terms of the monomials of these bases define we can express

$$I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2) = \sum_\alpha \overline{\mathcal{Q}}_\alpha(h_1)\overline{\mathcal{R}}_\alpha(h_2)$$

for some homogeneous polynomials $\overline{\mathcal{Q}}_\alpha$ on $\mathrm{II}(V_0)$ and $\overline{\mathcal{R}}_\alpha$ on $\mathrm{II}(W_0)$.

Define new polynomials $\mathcal{Q}_\alpha$ on $\mathrm{II}(W_0)$ and $\mathcal{R}_\alpha$ on $\mathrm{II}(V_0)$ by

$$\begin{aligned}
\mathcal{Q}_\alpha(h_1) &= \frac{1}{\mathrm{Vol}(K(V_0))} \int_{K(V_0)} \overline{\mathcal{Q}}_\alpha(ah_1)\, \Omega_{K(V_0)}(a) \\
\mathcal{R}_\alpha(h_2) &= \frac{1}{\mathrm{Vol}(K(W_0))} \int_{K(W_0)} \overline{\mathcal{R}}_\alpha(bh_2)\, \Omega_{K(W_0)}(b)
\end{aligned}$$

Then clearly $\mathcal{Q}_\alpha$ is invariant under $K(V_0)$, $\mathcal{R}_\alpha$ is invariant under $K(W_0)$, and both $\mathcal{Q}_\alpha$ and $\mathcal{R}_\alpha$ are homogeneous polynomials. Using the invariance properties of $I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2)$ given by part (3) of the last lemma

$$\begin{aligned}
&I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2) \\
&\quad = \frac{1}{\mathrm{Vol}(K(V_0))\,\mathrm{Vol}(K(W_0))} \iint_{K(V_0)\times K(W_0)} I_K^{\mathcal{P}}(V_0, ah_1, W_0, bh_1)\, \Omega_{K(V_0)\times K(W_0)}(a, b) \\
&\quad = \frac{1}{\mathrm{Vol}(K(V_0))\,\mathrm{Vol}(K(W_0))} \sum_\alpha \int \overline{\mathcal{Q}}_\alpha(ah_1)\, \Omega_{K(V_0)}(a) \int_{K(W_0)} \overline{\mathcal{R}}_\alpha(bh_2)\, \Omega_{K(W_0)}(b) \\
&\quad = \sum_\alpha \mathcal{Q}_\alpha(h_1)\mathcal{R}_\alpha(h_2).
\end{aligned}$$

Equation (6-5) now implies $\mathrm{degree}(\mathcal{Q}_\alpha)+\mathrm{degree}(R_\alpha) = l$ for each α. This completes the proof.

6.8 We now turn to the algebraic results needed for the proof of the transfer principle. The notation is similar to that of paragraph 4.12. That is let T' be another real inner product space of the same dimension as T, K' another compact Lie group of the same dimension as K with an orthogonal representation $a \mapsto a_*$ on T', $\rho : K \to K'$ a smooth isomorphism and $\psi : T \to T'$ a linear isomorphism that satisfy the conditions (a) and (b) of paragraph 4.12 (with $T(G/K)_{\mathbf{o}}$ replaced by T, $T(G'/K')_{\mathbf{o}'}$ by T' etc). Also, as before, we denote the isomorphism induced by ρ from objects defined on T to objects defined on T' by putting primes on the object in question. For example if $h_1 \in \mathrm{II}(V_0)$, $h_2 \in \mathrm{II}(W_0)$ and $H' \in \mathrm{EII}(T')$ then $h_1' \in \mathrm{II}(V_0')$, $h_2' \in \mathrm{II}(W_0')$ and $H' \in \mathrm{EII}(T')$ are given by

$$\begin{aligned}
h_1'(u, v) &= \psi h_1(\psi^{-1}u, \psi^{-1}v) \\
h_2'(u, v) &= \psi h_2(\psi^{-1}u, \psi^{-1}v) \\
H'(u, v) &= \psi H(\psi^{-1}u, \psi^{-1}v).
\end{aligned}$$

If $\mathcal{P}$ is a polynomial on $\mathrm{EII}(T)$, $\mathcal{Q}$ is a polynomial on $\mathrm{EII}(V_0)$ and $\mathcal{R}$ is a polynomial on $\mathrm{II}(W_0)$ then the polynomials $\mathcal{P}'$ (on $\mathrm{EII}(T')$), $\mathcal{Q}'$ (on $\mathrm{II}(V_0')$) and $\mathcal{R}'$ (on $\mathrm{II}(W_0')$) are given by

$$\begin{aligned} \mathcal{P}'(H') &= \mathcal{P}(H) && \text{all} && H \in \mathrm{EII}(T), \\ \mathcal{Q}'(h_1') &= \mathcal{Q}(h_1) && \text{all} && h_1 \in \mathrm{II}(V_0), \\ \mathcal{R}'(h_2') &= \mathcal{R}(h_2) && \text{all} && h_2 \in \mathrm{II}(W_0). \end{aligned}$$

The condition 4.12(b) implies first that $K'(V_0') = \rho K(V_0)$ and $K'(W_0') = \rho K(W_0)$ and second that if $H \in \mathrm{EII}(T)$, $h_1 \in \mathrm{II}(V_0)$, $h_2 \in \mathrm{II}(W_0')$ then $(aH)' = \rho(a)H$ $(a \in K)$, $(ah_1)' = \rho(a)h_1'$ $(a \in K(V_0))$ $(ah_2)' = \rho(a)h'$ $(a \in K(W_0))$. Therefore if $\mathcal{P}$ is a polynomial on $\mathrm{EII}(T)$ then $\mathcal{P}$ is invariant under K iff $\mathcal{P}'$ is invariant under K', with similar statements about polynomials on $\mathrm{II}(V_0)$ invariant under $K(V_0)$ and polynomials on $\mathrm{II}(W_0)$ invariant under $K(W_0)$ holding.

By chasing through the definitions it can be verified that for all $b \in K$ with $V_0 + b_*^{-1}W_0 = T$ that $G_{b_*^{-1}W_0}(V_0, h_1)' = G_{\rho(b)_*^{-1}W_0'}(V_0', h_1')$ for all $h_1 \in \mathrm{II}(V_0)$ and that $\sigma(V_0^\perp, b_*^{-1}W_0^\perp) = \sigma(V_0'^\perp, \rho(b)_*^{-1}W_0'^\perp)$. Therefore from the change of variable $b \mapsto \rho(b)$ in the integral defining $I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2)$ it follows

$$I_{K'}^{\mathcal{P}'}(V_0', h_1', W_0', h_0') = I_K^{\mathcal{P}}(V_0, h_1, W_0, h_0)$$

Whence if

$$I_K^{\mathcal{P}}(V_0, h_1, W_0, h_0) = \sum_\alpha \mathcal{Q}_\alpha(h_1)\mathcal{R}_\alpha(h_2)$$

is the decomposition of $I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2)$ given by lemma 6.5 that

$$I_{K'}^{\mathcal{P}'}(V_0', h_1', W_0', h_0') = \sum_\alpha \mathcal{Q}_\alpha'(h_1')\mathcal{R}_\alpha'(h_2')$$

where the pairs $(\mathcal{Q}_\alpha', \mathcal{R}_\alpha')$ have all the properties listed for the pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ in lemma 6.5.

7. Proof of the kinematic formula and the transfer principle

7.1 We will use the notation in the statement of the kinematic formula 4.10. If $h_1 \in \mathrm{II}(V_0)$ and $h_2 \in \mathrm{II}(W_0)$ then define $I_K^{\mathcal{P}}(V_0, h_1, W_0, h_2)$ as in definition 6.3. Let M be a submanifold of G/K of type V_0 and $x \in M$. Then there is a $\xi \in G$ with $\xi_* V_0 = TM_x$ and if ξ_1 is any other element of G with $\xi_{1*}V_0 = TM_x$ then $\xi_1 = \xi a$ for some $a \in K(V_0)$, both $h_{\mathbf{o}}^{\xi^{-1}M}$ and $h_{\mathbf{o}}^{\xi_1^{-1}M}$ are in $\mathrm{II}(V_0)$ and $h^{\xi_1^{-1}M} = a^{-1}h^{\xi^{-1}M}$. Likewise if N is a submanifold of G/K of type W_0, η, η_1 elements of G with $\eta_* W_0 = TN_y$ and $\eta_{1*}W_0 = TN_y$ then $\eta_1 = b\eta$ for some $b \in K(W_0)$. Therefore by the invariance properties of $I_K^{\mathcal{P}}$ given by lemma 6.4 part (3) it follows that if we define

$$I_K^{\mathcal{P}}(V_0, h_x^M, W_0, h_y^N) = I_K^{\mathcal{P}}(V_0, h_x^{\xi^{-1}M}, W_0, h_y^{\eta^{-1}N}), \qquad \xi_* V_0 = TM_x, \quad \eta_* W_0 = TN_y$$

then this definition is independent of the choice of ξ with $\xi_* V_0 = TM_x$ and η with $\eta_* W_0 = TNy$. We can now give a statement of a kinematic formula which, together with what has already been done, easily implies our first statement of the kinematic formula and the transfer principle.

7.2 Kinematic formula. *With the notation and hypothesis of 4.10 the kinematic formula*

$$\int_G I^{\mathcal{P}}(M \cap gN)\,\Omega_G(g) = \iint_{M\times N} I^{\mathcal{P}}_K(V_0, h^M_x, W_0, h^N_y)\,\Omega_{M\times N}(x,y) \tag{7-1}$$

holds.

7.3 Proof of 4.10 and 4.12. Once the formula just given has been proven then theorem 4.10 follows at once from lemma 6.5 and the transfer principle 4.12 follows from the results in paragraph 6.8.

7.4 Proof of 7.2. Let $\widehat{M} = \pi^{-1}M$ and $\widehat{N} = \pi^{-1}N$ where $p : G \to G/K$ is the natural projection. Define $h : \widehat{M} \times \widehat{N} \to \mathbb{R}$ by

$$h(\xi, \eta) = \mathcal{P}(H^{M\cap \xi\eta^{-1}N}_{\pi\xi}) \tag{7-2}$$

when M and $\xi\eta^{-1}N$ intersect transversely at $\pi\xi$ and $h(\xi,\eta) = 0$ otherwise. Using this function h and the submanifolds $\widehat{M}$ and $\widehat{N}$ of G in the basic integral formula of paragraph 2.7 (and recalling the assumption $\Delta \equiv 1$)

(7-3)

$$\int_G \left(\int_{\widehat{M}\cap g\widehat{N}} h\circ\varphi_g\,\Omega_{\widehat{M}\cap g\widehat{N}}\right)\Omega_G(g) = \iint_{\widehat{M}\times\widehat{N}} h(\xi,\eta)\sigma(T^\perp\widehat{M}_\xi, T^\perp\widehat{N}_\eta)\,\Omega_{\widehat{M}\times\widehat{N}}(\xi,\eta)$$

Let g be an element of G so that $\widehat{M}$ and $g\widehat{N}$ intersect transversely, which is equivalent to having M and gN intersecting transversely as π is a Riemannian submersion. Let $\xi \in \widehat{M} \cap g\widehat{N}$. Then using the definition of φ_g

$$h\circ\varphi_g(\xi) = h(\xi, g^{-1}\xi) = \mathcal{P}(H^{M\cap\xi\xi^{-1}gN}_{\pi\xi}) = \mathcal{P}(H^{M\cap gN}_{\pi\xi}) \tag{7-4}$$

Therefore if $\pi(\xi_1) = \pi(\xi) = x \in M \cap gN$ then $h\circ\varphi_g(\xi) = h\circ\varphi_g(\xi_1) = \mathcal{P}(H^{M\cap gN}_x)$. Whence the restriction of π to $\widehat{M}\cap g\widehat{N}$ is a Riemannian submersion that fibers with fibres isometric to K and the function $h\circ\varphi_g$ has the constant value $\mathcal{P}(H^{M\cap gN}_x)$ on the fibre over x. Whence

$$\begin{aligned}\int_{\widehat{M}\cap g\widehat{N}} h\circ\varphi_g\,\Omega_{\widehat{M}\cap\widehat{N}} &= \mathrm{Vol}(K)\int_{M\cap gM} \mathcal{P}(H^{M\cap gN}_x)\,\Omega_{M\cap gN}(x)\\ &= \mathrm{Vol}(K) I^{\mathcal{P}}(M\cap gN)\end{aligned} \tag{7-5}$$

for all g for which M and gN intersect transversely and by theorem 2.7 this is the case for almost all $g \in G$. Thus integration with respect to g yields

$$\int_G \left(\int_{\widehat{M}\cap g\widehat{N}} h\circ\varphi_g\,\Omega_{\widehat{M}\cap g\widehat{N}}\right)\Omega_G(g) = \mathrm{Vol}(K)\int_G I^{\mathcal{P}}(M\cap gN)\,\Omega_G(g) \tag{7-6}$$

provided the integral on the left converges, a consideration we will return to shortly.

Returning to equation (7-3) we use that the map $(\xi,\eta)\mapsto(\pi\xi,\pi\eta)$ is a Riemannian submersion of $\widehat{M}\times\widehat{N}$ onto $M\times N$ which fibers $\pi^{-1}[x]\times\pi^{-1}[y]$ isometric to $K\times K$. Therefore, at least formally,

$$\begin{aligned}
&\iint_{\widehat{M}\times\widehat{N}} h(\xi,\eta)\sigma(T^\perp\widehat{M}_\xi,T^\perp\widehat{N}_\eta)\,\Omega_{\widehat{M}\times\widehat{N}}(\xi,\eta)\\
&=\iint_{M\times N}\left(\iint_{\pi^{-1}[x]\times\pi^{-1}[y]} h(\xi,\eta)\sigma(T^\perp\widehat{M}_\xi,T^\perp\widehat{N}_\eta)\,\Omega_{\pi^{-1}[x]\times\pi^{-1}[y]}(\xi,\eta)\right)\Omega_{M\times N}(x,y)\\
&=\iint_{M\times N}\mathcal{I}(x,y)\,\Omega_{M\times N}(x,y) \qquad (7\text{-}8)
\end{aligned}$$

with

$$\mathcal{I}(x,y)=\iint_{\pi^{-1}[x]\times\pi^{-1}[y]} h(\xi,\eta)\sigma(T^\perp\widehat{M}_\xi,T^\perp\widehat{N}_\eta)\,\Omega_{\pi^{-1}[x]\times\pi^{-1}[y]}(\xi,\eta) \qquad (7\text{-}9)$$

Also define

$$|\mathcal{I}|(x,y)=\iint_{\pi^{-1}[x]\times\pi^{-1}[y]} |h(\xi,\eta)|\sigma(T^\perp\widehat{M}_\xi,T^\perp\widehat{N}_\eta)\,\Omega_{\pi^{-1}[x]\times\pi^{-1}[y]}(\xi,\eta)$$

We will now show that integral defining $|\mathcal{I}|(x,y)$ converges and that the function $(x,y)\mapsto|\mathcal{I}|(x,y)$ on $M\times N$ is bounded. Putting this into equation (7-7) and using Fubini's theorem (or rather its generalization to the present context see §33 of [15]) will show that the left side of (7-7) and therefore the right side of (7-3) converges absolutely. By the basic integral formula this is enough to guarantee the convergence of all our integrals.

First use that M is of type V_0 and N of type W_0 to choose for each $x\in M$ and $y\in N$ elements $\xi_x,\eta_y\in G$ such that

$$\xi_{x*}V_0=TM_x \quad\text{and}\quad \eta_{y*}W_0=TN_y \qquad (7\text{-}10)$$

Then $(a,b)\mapsto(\xi_x a,\eta_y b)$ is an isometry of $K\times K$ with $\pi^{-1}[x]\times\pi^{-1}[y]$ and thus doing a change of variable in (7-9)

$$|\mathcal{I}|(x,y)=\iint_{K\times K}|h(\xi_x a,\eta_y b)|\sigma(T^\perp\widehat{M}_{\xi_x a},T^\perp\widehat{N}_{\eta_y b})\,\Omega_{K\times K}(a,b) \qquad (7\text{-}11)$$

By equation (3-13)

$$\begin{aligned}
\sigma(T^\perp\widehat{M}_{\xi_x a},T^\perp\widehat{N}_{\eta_y b})&=\sigma(a_*^{-1}\xi_{x*}^{-1}T^\perp M_x,b_*^{-1}\eta_{y*}^{-1}T^\perp N_y)\\
&=\sigma(a_*^{-1}V_0^\perp,b_*^{-1}W_0^\perp)\\
&=\sigma(V_0^\perp,(ab^{-1})_*W_0^\perp) \qquad (7\text{-}12)
\end{aligned}$$

By the estimate (6-5) (with (V, h_1) replaced by $(V_0, h_{\mathbf{o}}^{\xi_x^{-1}M})$ and $(b_*^{-1}W, b^{-1}W_0)$ replaced by $((ab^{-1})_* W_0, (ab^{-1})h_{\mathbf{o}}^{\eta^{-1}N})$,

$$
\begin{aligned}
|h(\xi_x a, \eta_y b)| &= |\mathcal{P}(H^{M\cap \xi a(\eta b)^{-1}N})| = |\mathcal{P}(H_{\mathbf{o}}^{\xi^{-1}Mab^{-1}\eta^{-1}N})| \\
&= \Big|\mathcal{P}\Big(G_{(ab^{-1})_*\eta_*^{-1}TN_y}(\xi_*^{-1}TM_x, h_{\mathbf{o}}^{\xi^{-1}M}) \\
&\qquad + G_{\xi_*^{-1}TM_x}((ab^{-1})_* TN_y, (ab^{-1})h_{\mathbf{o}}^{\eta^{-1}N})\Big)\Big| \\
&= \Big|\mathcal{P}\Big(G_{(ab^{-1})_*W_0}(V_0, h_{\mathbf{o}}^{\xi^{-1}M}) + G_{V_0}((ab^{-1})_*W_0, (ab^{-1})h_{\mathbf{o}}^{\eta^{-1}N})\Big)\Big| \\
&\le C(\mathcal{P})\Big(\|h_{\mathbf{o}}^{\xi^{-1}M}\|^2 + \|h_{\mathbf{o}}^{(ab^{-1})\eta^{-1}N}\|^2\Big)^{\frac{l}{2}}\,\sigma(V_0^{\perp}, (ab^{-1})_*W_0^{\perp})^{-l} \\
&= C(\mathcal{P})(\|h_x^M\|^2 + \|h_y^N\|^2)^{\frac{l}{2}}\sigma(V_0^{\perp}, (ab^{-1})_*W_0^{\perp})^{-l} \qquad (7\text{-}13)
\end{aligned}
$$

The function $(x, y) \mapsto (\|h_x^M\|^2 + \|h_y^N\|^2)^{l/2}$ is continuous on the compact space $M \times N$ so there is a constant B with $B \ge (\|h_x^M\|^2 + \|h_y^N\|^2)^{l/2}$ for all x, y. Using this bound, equation (7-12), equation (7-13) and a change of variable in (7-9)

$$
\begin{aligned}
|\mathcal{I}|(x, y) &\le C(\mathcal{P})B\int_K\int_K \sigma(V_0^{\perp}, (ab^{-1})_*W_0^{\perp})^{1-l}\,\Omega_K(b)\,\Omega_K(a) \\
&= C(\mathcal{P})B\,\mathrm{Vol}(K)\int_K \sigma(V_0^{\perp}, b_*^{-1}W_0^{\perp})^{1-l}\,\Omega_K(b).
\end{aligned}
$$

This integral converges by premise (b) of 4.10. Therefore the integral defining $|\mathcal{I}|(x, y)$ converges and the last inequality gives an upper bound for $|\mathcal{I}|(x, y)$ that holds on all of $M \times N$. This verifies our claims about $|\mathcal{I}|(x, y)$ and shows that all our integrals converge.

If the absolute values are removed from the first several lines of (7-13) the calculation still holds. Thus,

$$
h(\xi_x a, \eta_y b) = \mathcal{P}(G_{(ab^{-1})_*W_0}(V_0, h_{\mathbf{o}}^{\xi_x^{-1}M}) + G_{V_0}((ab^{-1})_*W_0, (ab^{-1})h_{\mathbf{o}}^{\eta_y^{-1}N})).
$$

Using this equation, equation (7-12), and expressing $\mathcal{I}(x, y)$ as an integral over $K \times K$ instead of over $\pi^{-1}[x] \times \pi^{-1}[y]$ (just as was done with $|\mathcal{I}|(x, y)$ in equation (7-11)) we find

$$
\begin{aligned}
\mathcal{I}(x, y) &= \int_K \Big(\int_K h(\xi_x a, \eta_y b)\sigma(T^{\perp}\widehat{M}_{\xi_x a}, T^{\perp}\widehat{N}_{\eta_y b})\,\Omega_K(b)\Big)\,\Omega_K(a) \\
&= \int_K \Big(\int_K \mathcal{P}\Big(G_{(ab^{-1})_*W_0}(V_0, h_{\mathbf{o}}^{\xi_x^{-1}M}) \\
&\qquad + G_{V_0}((ab^{-1})_*W_0, (ab^{-1})h_{\mathbf{o}}^{\eta_y^{-1}N})\Big)\sigma(V_0^{\perp}, b_*^{-1}W_0^{\perp})\,\Omega_K(b)\Big)\,\Omega_K(a) \\
&= \int_K \Big(\int_K \mathcal{P}\Big(G_{b_*^{-1}W_0}(V_0, h_{\mathbf{o}}^{\xi_x^{-1}M}) \\
&\qquad + G_{V_0}(b_*^{-1}W_0, b^{-1}h^{\eta_y^{-1}N})\Big)\sigma(V_0^{\perp}, b_*^{-1}W_0^{\perp})\,\Omega_K(b)\Big)\,\Omega_K(a) \\
&= \int_K I_K^{\mathcal{P}}(V_0, h^{\xi_x^{-1}M}, W_0, h^{\eta_y^{-1}N})\,\Omega_K(a) \\
&= \mathrm{Vol}(K)\,I_K^{\mathcal{P}}(V_0, h_x^M, W_0, h_y^N).
\end{aligned}
$$

Putting this into equation (7-7) and putting the result of that and also equation (7-6) into equation (7-3) completes the proof of theorem 7.2.

8. Spaces of Constant Curvature

8.1 In this section G/K will be assumed to be the simply connected manifold of constant sectional curvature c and dimension n and G is the full isometry group of G/K. The case where G is the group of all orientation preserving isometries of G/K can be dealt with in the same manner, the details in this case are left to the reader. If $O(T(G/K)_{\mathbf{o}})$ is the orthogonal group of the inner product space $T(G/K)_{\mathbf{o}}$ then the map $a \mapsto a_*$ gives a smooth isomorphism of K with $O(T(G/K)_{\mathbf{o}})$ and we will identify K with $O(T(G/K)_{\mathbf{o}})$ via this isomorphism. As in paragraph 3.12 example (a) we normalize so that $\mathrm{Vol}(K) = \mathrm{Vol}(O(n)) = 2\,\mathrm{Vol}(SO(n))$. With the identification we have just made of K with $O(T(G/K)_{\mathbf{o}})$ it is easy to check that if V_0 is any p dimensional subspace of $T(G/K)_{\mathbf{o}}$ that every p dimensional submanifold of G/K is of type V_0 (in the sense of definition 4.2) and

$$K(V_0) = O(V_0) \times O(V_0^{\perp}) \tag{8-1}$$

where $O(V_0)$ and $O(V_0^{\perp})$ are the orthogonal groups on V_0 and $V_0^{\perp}$ respectively.

8.2 Therefore the general kinematic formula 4.10 can be restated in this case. Let V_0 be a p dimensional and W_0 a q dimensional subspace of $T(G/K)_{\mathbf{o}}$ and let $\mathcal{P}$ be a homogeneous polynomial of degree l on $\mathrm{EII}(T(G/K)_{\mathbf{o}})$ which is invariant under $O(T(G/K)_{\mathbf{o}})$ and such that

$$l \le p + q - n + 1 \tag{8-2}$$

Then there is a finite set of pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ such that

(1) each $\mathcal{Q}_\alpha$ is a homogeneous polynomial on $\mathrm{II}(V_0)$ invariant under $O(V_0) \times O(V_0)$,
(2) each $\mathcal{R}_\alpha$ is a homogeneous polynomial on $\mathrm{II}(V_0)$ invariant under $O(W_0) \times O(W_0)$,
(3) $\mathrm{degree}(\mathcal{Q}_\alpha) + \mathrm{degree}(\mathcal{R}_\alpha) = l$ for each α and
(4) for all compact p dimensional submanifolds M and compact q dimensional submanifolds N of G/K (they may have boundary)

$$\int_G I^{\mathcal{P}}(M \cap gN)\,\Omega_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha} I^{\mathcal{R}_\alpha}(N).$$

8.3 Since the invariant theory of orthogonal groups is well understood (for example see [22]) for a given degree k it is not hard to list all the invariant polynomials on $\mathrm{II}(V_0)$ or $\mathrm{II}(W_0)$ that are homogeneous of degree k and invariant under $O(V_0) \times O(V_0)$ or $O(W_0) \times O(W_0)$. We will give such a list for $k \le 4$ below.

8.4 To prove the results of paragraph 8.2 from theorem 4.10 it only remains to show the inequality (8-2) implies the inequality

$$\int_{O(T)} \sigma(V_0^{\perp}, aW_0^{\perp})^{1-l}\,\Omega_{O(T)}(a) < \infty \tag{8-3}$$

where we have set $T = T(G/K)_{\mathbf{o}}$ for brevity. To do this we follow Chern [6]. To start let $G_q(T)$ be the Grassmann manifold of all q planes in T. Then there is a submersion of $O(T)$ onto $G_q(T)$ given by $a \mapsto aV_0$. Put the Riemannian metric on $G_q(T)$ that makes this map into a Riemannian submersion. Then this map fibers with fibres isometric with $O(W_0) \times O(W_0^{\perp})$ and s constant on each fibre, whence to prove (8-3) it is enough to prove

$$\int_{G_q(T)} \sigma(V_0^{\perp}, W^{\perp})^{1-l}\, \Omega_{G_q(T)}(W) < \infty \tag{8-4}$$

We will show this by adapting a formula of Chern's to the present case. Unfortunately this involves some excess notation. First Let $G_q^*(T)$ be the subset of $G_q(T)$ of all q planes W with $\dim(V_0 \cap W) = p + q - n$. Then $G_q^*(T)$ differs from $G_q(T)$ by a set of measure zero whence integrals over $G_q(T)$ can be replaced by integrals over $G_q^*(T)$.

Let $G_{p+q-n}(V_0)$ be the Grassmann manifold of all $p+q-n$ planes in V_0 with its standard metric invariant under $O(V_0)$. For each $U \in G_{p+q-n}(V_0)$ let $G_q(U,T)$ be the set of all q planes W in T with $U \subseteq W$. Then there is a natural bijection between $G_q(U,T)$ and the Grassmann manifold of $q-(p+q-n) = n-p$ planes in the quotient space T/U, an $n-(p+q-n) = 2n-p-q$ dimensional vector space. Give $G_q(U,T)$ the metric that makes this bijection an isometry of $G_q(U,T)$ with $G_{n-p}(T/U)$. Let $G_q^*(U,T)$ be the subset of $G_q(U,T)$ of q planes W with $W \cap V_0 = U$. Then if $\pi : G_p^*(T) \to G_{p+q-n}(V_0)$ is given by $\pi(W) = V_0 \cap W$ then $\pi^{-1} = G_q(U,T)$ and this differs from $G_q^*(U,T)$ by a set of measure zero, thus integrals over $G_q(U,T)$ can be replaced by integrals over $G_q^*(U,T)$.

Chern proved (equation (28) of section 2 in [6]) the equality of densities

$$\begin{aligned} \Omega_{G_q(T)}(W) &= \sigma(V_0^{\perp}, W^{\perp})^{p+q-n}\Omega_{G_q(\pi W,T)} \wedge \pi^*\Omega_{G_{p+q-n}(V_0)} \\ &= \sigma(V_0^{\perp}, W^{\perp})^{p+q-n}\Omega_{G_q(W\cap V_0,T)} \wedge \pi^*\Omega_{G_{p+q-n}(V_))} \end{aligned} \tag{8-5}$$

which holds for all $W \in G_q^*(T)$. This is also proven in Santaló's book [18] (equation 14.40 on page 241) where the notation is a little closer to that used here.

It follows by the lemma on fibre integration in the appendix that if h is any measurable function defined almost everywhere on $G_q(T)$ that

$$\begin{aligned} &\int_{G_q(T)} h(W)\, \Omega_{G_p(T)}(W) \\ &= \int_{G_{p+q-n}(V_0)} \int_{W\cap V_0 = U} h(W)\sigma(V_0^{\perp}, W^{\perp})^{p+q-n}\, \Omega_{G_q(U,T)}(W)\, \Omega_{G_{p+q-n}(V_0)}(U) \end{aligned} \tag{8-6}$$

To prove the inequality (8-4) let $h(W) = \sigma(V_0,^{\perp}, W^{\perp})^{1-l}$ in (8-6) and use that $l \leq p+q-n+1$ so that the function $W \mapsto h(W)\sigma(V_0^{\perp}, W^{\perp})^{p+q-n}$ is bounded. This completes the proof.

8.5 We now give a list of the homogeneous polynomials on $\mathrm{II}(V_0)$ of small degree which are invariant under $O(V_0) \times O(V_0^{\perp})$. Choose an orthonormal basis $e_1, \ldots, e_n$

of $T = T(G/K)_{\mathbf{o}}$ such that $e_1, \dots, e_p$ is a basis of V_0 and $e_{p+1}, \dots, e_n$ is a basis of V_0. Then if $h \in \mathrm{II}(V_0)$ and $H \in \mathrm{EII}(T)$ define the components h and H by

$$\begin{aligned} h^\alpha_{ij} &= \langle h(e_i, e_j), e_\alpha \rangle \qquad 1 \le i, j \le p, \quad p+1 \le \alpha \le n \\ H^\alpha_{ij} &= \langle H(e_i, e_j), e_\alpha \rangle \qquad 1 \le i, j, \alpha \le n. \end{aligned} \tag{8-7}$$

Then there are no homogeneous polynomials of odd degree on $\mathrm{II}(V_0)$ invariant under $O(V_0) \times O(V_0)$ (as the polynomial must be invariant under $h \mapsto -h$). The polynomials homogeneous of degree 2 invariant under $O(V_0) \times O(V_0^\perp)$ are spanned by the two polynomials

$$\mathcal{Q}_1(h) = \sum_{i,j,\alpha} (h^\alpha_{ij})^2, \qquad \mathcal{Q}_2(h) = \sum_\alpha (\sum_i h^\alpha_{ii})^2 \tag{8-8}$$

and if $2 < p < n-1$ these polynomials are independent. Geometrically $\mathcal{Q}_1(h)$ is the length of the second fundamental form and $\mathcal{Q}_2(h)$ is p^2 times the square of the length of the mean curvature vector.

The homogeneous polynomials of degree four on $\mathrm{II}(V_0)$ invariant under $O(V_0) \times O(V_0)$ are spanned by the eight polynomials

$$\begin{aligned} \mathcal{R}_1 &= \sum h^\alpha_{ii} h^\alpha_{jj} h^\beta_{kk} h^\beta_{ll}, & \mathcal{R}_2 &= \sum h^a_{ii} h^\alpha_{jk} h^\beta_{jk} h^\beta_{ll} \\ \mathcal{R}_3 &= \sum h^\alpha_{ij} h^\alpha_{ij} h^\beta_{kk} h^\beta_{ll}, & \mathcal{R}_4 &= \sum h^\alpha_{ij} h^\alpha_{ij} h^\beta_{kl} h^\beta_{kl} \\ \mathcal{R}_5 &= \sum h^\alpha_{ij} h^\alpha_{ik} h^\beta_{jk} h^\beta_{ll}, & \mathcal{R}_6 &= \sum h^\alpha_{ij} h^\alpha_{ik} h^\beta_{jl} h^\beta_{kl} \\ \mathcal{R}_7 &= \sum h^\alpha_{ij} h^\alpha_{kl} h^\beta_{ij} h^\beta_{kl}, & \mathcal{R}_8 &= \sum h^\alpha_{ij} h^\alpha_{kl} h^\beta_{ik} h^\beta_{jl} \end{aligned}$$

and these are linearly independent provided $4 \le p \le n-2$. To find these write the expression $h^{\alpha_1}_{1_1 i_2} h^{\alpha_2}_{i_3 i_4} h^{\alpha_3}_{i_5 i_6} h^{\alpha_4}_{i_7 i_8}$ and contract the α's in pairs and then the i's in pairs. For example contracting α_1 with α_2, α_3 with α_4, i_1 with i_2, i_3 with i_4, i_5 with i_6 and i_7 with i_8 leads to $\mathcal{R}_1$ above. See [22] for details.

Having this list of invariants does make one thing clear, that every polynomial $\mathcal{P}$ on $\mathrm{II}(V_0)$ invariant under $O(V_0) \times O(V_0^\perp)$ is a restriction of a polynomial $\widehat{\mathcal{P}}$ on $\mathrm{EII}(T)$ which is invariant under $O(T)$. (Where we view $h \in \mathrm{EII}(T)$ as an element of $\mathrm{EII}(T)$ by setting $H(u,v) = h(Pu, Pv)$ where $P : T \to V_0$ is the orthogonal projection and $u, v \in T$.) To see this suppose that $\mathcal{P} = \mathcal{R}_3$ in the list above. Then define $\widehat{\mathcal{P}}$ on $\mathrm{EII}(T)$ by

$$\widehat{\mathcal{P}} = \sum H^\alpha_{ij} H^\alpha_{ij} H^\beta_{kk} H^\beta_{ll}$$

where this time we sum over the range of indices $1 \le i, j, k, k, a, b \le n$ instead of $1 \le i, j, k, l \le p$, $p+1 \le \alpha, \beta \le n$.

8.6 We now give an example of how to use theorem 4.10 and the invariant theory of the isotropy subgroup to prove a kinematic formula. For each k with $2 \le k \le n-1$ let $\mathcal{U}_k$ be the invariant polynomial defined on the second fundamental forms of k dimensional submanifolds of $\mathbb{R}^n$ by

$$\begin{aligned} \mathcal{U}_k(h) &= k\mathcal{Q}_1(h) - \mathcal{Q}_2(h) \\ &= k \sum_{i,j,\alpha} (h^\alpha_{ij})^2 - \sum_\alpha \big(\sum_i h^\alpha_{ij}\big)^2, \end{aligned}$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the invariant polynomials defined in equation (8-8). If $S^k(r) \subset \mathbb{R}^{k+1} \subset \mathbb{R}^n$ is the standard imbedding of a k dimensional sphere of radius r into $\mathbb{R}^n$ then $\mathcal{U}_k\left(h_x^{S^k(r)}\right) = 0$ for all $x \in S^k(r)$.

The invariant polynomials on the second fundamental forms of p dimensional submanifolds has as a basis $\mathcal{U}_p$ and $\mathcal{Q}_1$. Likewise the invariant polynomials on the second fundamental forms of q dimensional submanifolds has as a basis the polynomials $\mathcal{U}_q$ and $\mathcal{Q}_1$. Thus by the general kinematic formula given by theorem 4.10 there are constants c_i so that if G/K is as in the first paragraph of this section and $p+q-n \geq 2$,

$$\begin{aligned}&\int_G I^{\mathcal{U}_{p+q-n}}(M^p \cap gN^q)\,\Omega_G(g)\\&\quad= \left(c_1 I^{\mathcal{U}_p}(M^p) + c_2 I^{\mathcal{Q}_1}(M^p)\right)\operatorname{Vol}(N^q) + \operatorname{Vol}(M^p)\left(c_3 I^{\mathcal{U}_q}(N^q) + c_4 I^{\mathcal{Q}_1}(N^q)\right)\end{aligned}$$

If we now assume that $G/K = \mathbb{R}^n$, $M^p = S^p \subset \mathbb{R}^{p+1} \subset \mathbb{R}^n$ is the standard way and N^q is a bounded domain in $\mathbb{R}^q \subset \mathbb{R}^n$, then for almost every $g \in G$ the intersection $M^p \cap gN^q$ is either empty or congruent to a standard imbedding of a sphere $S^{p+q-n}(r)$. Thus in this case $\mathcal{U}_{p+q-n}(h^{M^p \cap gN^q}) \equiv 0$ for almost all $g \in G$. Also $h^{N^q} \equiv 0$, so that $\mathcal{U}_q(h^{N^q}) \equiv 0$ and $\mathcal{Q}_1(h^{N^q}) \equiv 0$. But in this case $I^{\mathcal{Q}_1}(M^q) \neq 0$. Using this in the last equation shows that $c_2 = 0$. A similar trick shows that $c_4 = 0$. This and the transfer principle lead to:

Proposition. *There are constants $c(p,q,n)$ $(p+q-n \geq 2)$ so that for any compact submanifolds (possibly with boundary) M^p and N^q of a space G/K of constant sectional curvature with K isomorphic to $O(n)$ and normalized so that* $\operatorname{Vol}(K) = \operatorname{Vol}(O(n))$ *the kinematic formula*

$$\int_G I^{\mathcal{U}_{p+q-n}}(M^p \cap gN^q)\,\Omega_G(g) = c(p,q,n) I^{\mathcal{U}_p}(M^p)\operatorname{Vol}(N^q) + c(q,p,n)\operatorname{Vol}(M^p) I^{\mathcal{U}_q}(N^q)$$

holds.

We close this section with a more concrete application of the transfer principle to three dimensional spaces of constant curvature. In his paper [3] C.-S. Chen proved that if M and N are compact surfaces in $\mathbb{R}^3$ and G is the group of orientation preserving isometries of $\mathbb{R}^3$ (with the same normalizations used in example (a) of paragraph 3.12) that the following very pretty formula holds

$$\begin{aligned}&\int_G \int_{M\cap gN} \kappa^2 ds\,\Omega_G(g)\\&\qquad= \pi^3 \operatorname{Area}(M)\int_N (2H^2 + \|h\|^2)\,\Omega_N + \pi^3 \operatorname{Area}(N)\int_M (2H^2 + \|h\|^2)\,\Omega_M\end{aligned}$$

where κ is the curvature of the curve $M \cap gN$, $H^2 = ((\lambda_1+\lambda_2)/2)^2$ is the square of the mean curvature, and $\|h\|^2 = \lambda_1^2 + \lambda_2^2$ is the square length of the second fundamental form (here λ_1 and λ_2 are the principle curvatures). By the transfer principle this formula holds exactly in the form written for all compact surfaces in any simply connected space of constant sectional curvature.

9. An algebraic characterization of the polynomials in the Weyl tube formula.

9.1 In this section we define the polynomials in the components of the second fundamental form of a submanifold that appear in the Weyl tube formula (this formula will be stated and its proof sketched in the next section). These polynomials will be characterized as the unique invariant polynomials which vanish on the second fundamental forms of "generalized cylinders". In the next section we will show this characterization can be used to give an easy proof of Weyl's formula and an elementary proof of the kinematic formula of Chern and Federer from theorem 7.2 above.

9.2 For the rest of this section we will use the notation introduced in paragraph 6.1, that is T is an n dimensional real inner product space, V_0 a p dimensional subspace of T etc. If $h \in \mathrm{II}(V_0)$ and $H \in \mathrm{EII}(V_0)$ then the components of h and H are defined by equation (8-7) above. Define $R_{ij}^{st}(h)$ and $R_{ij}^{st}(H)$ by

$$R_{ij}^{st}(h) = \sum_{\alpha=p+1}^{n} (h_{is}^{\alpha} h_{jt}^{\alpha} - h_{it}^{\alpha} h_{js}^{\alpha}) \qquad 1 \le i,j,s,t \le p \tag{9-1}$$

$$R_{ij}^{st}(H) = \sum_{\alpha=1}^{n} (H_{is}^{\alpha} H_{jt}^{\alpha} - H_{it}^{\alpha} H_{js}^{\alpha}) \qquad 1 \le i,j,s,t \le n \tag{9-2}$$

If h is the second fundamental form of a submanifold of Euclidean space then the Gauss curvature equation implies that $R_{ij}^{st}(h)$ are the components of the curvature tensor of the submanifold.

$$\begin{aligned}
w_{2l}(h) &= \sum_{\substack{1\le i_1,\dots,i_{2l}\le p\\ 1\le j_1,\dots,j_{2l}\le p}} \delta_{j_1\cdots j_{2l}}^{i_1\cdots i_{2l}} R_{i_1 i_2}^{j_1 j_2}(h)\cdots R_{i_{2l-1} i_{2l}}^{j_{2l-1} j_{2l}}(h)\\
&= 2^l \sum_{\substack{1\le i_1,\dots,i_{2l}\le p\\ 1\le j_1,\dots,j_{2l}\le p\\ p+1\le \alpha_1,\dots,\alpha_l\le n}} \delta_{j_1\cdots j_{2l}}^{i_1\cdots i_{2l}} h_{i_1 j_1}^{\alpha_1} h_{i_2 j_2}^{\alpha_1} \cdots h_{i_{2l-1} j_{2l-1}}^{\alpha_l} h_{i_{2l} j_{2l}}^{\alpha_l}\\
&= 2^l \sum_{\substack{1\le i_1,\dots,i_{2l}\le p\\ 1\le j_1,\dots,j_{2l}\le p\\ p+1\le \alpha_1,\dots,\alpha_l\le n}} \delta_{j_1\cdots j_{2l}}^{i_1\cdots i_{2l}} \prod_{t=1}^{l} (h_{i_{2t-1} j_{2t-1}}^{\alpha_t} h_{i_{2t} j_{2t}}^{\alpha_t})\\
&= 2^l \sum_{\substack{1\le i_1,\dots,i_{2l}\le p\\ p+1\le \alpha_1,\dots,\alpha_l\le n}} \det \begin{bmatrix} h_{i_1 i_1}^{\alpha_1} & h_{i_1 i_2}^{\alpha_1} & \cdots & h_{i_1 i_{2l}}^{\alpha_1}\\ h_{i_2 i_1}^{\alpha_1} & h_{i_2 i_2}^{\alpha_1} & \cdots & h_{i_2 i_{2l}}^{\alpha_2}\\ \vdots & \vdots & \ddots & \vdots\\ h_{i_{2l-1} i_1}^{\alpha_l} & h_{i_{2l-1} i_2}^{\alpha_l} & \cdots & h_{i_{2l-1} i_{2l}}^{\alpha_l}\\ h_{i_{2l} i_1}^{\alpha_l} & h_{i_{2l} i_2}^{\alpha_l} & \cdots & h_{i_{2l} i_{2l}}^{\alpha_l} \end{bmatrix}
\end{aligned} \tag{9-3}$$

where $\delta_{j_1\dots j_k}^{i_1\dots i_k}$ is the generalized Kronecker delta which is zero unless $i_1, \dots, i_k$ are all distinct and $j_1, \dots, j_k$ are all distinct and $j_1, \dots, j_k$ is a permutation of $i_1, \dots, i_k$ in which case it is the sign of this permutation. By definition when $l = 0$

$$w_0 = 1$$

Also define polynomials on $\mathrm{EII}(T)$, also denoted by "w_{2l}", by

$$
\begin{aligned}
w_{2l}(H) &= \sum_{\substack{1\le i_1,\dots,i_{2l}\le n\\ 1\le j_1,\dots,j_{2l}\le n}} \delta^{i_1\dots i_{2l}}_{j_1\dots j_{2l}} R^{j_1j_2}_{i_1i_2}(H)\cdots R^{j_{2l-1}j_{2l}}_{i_{2l-1}i_{2l}}(H)\\
&= 2^l \sum_{\substack{1\le i_1,\dots,i_{2l}\le n\\ 1\le \alpha_1,\dots,\alpha_l\le n}} \det\begin{bmatrix} H^{\alpha_1}_{i_1i_1} & H^{\alpha_1}_{i_1i_2} & \dots & H^{\alpha_1}_{i_1i_{2l}}\\ H^{\alpha_1}_{i_2i_1} & H^{\alpha_1}_{i_2i_2} & \dots & H^{\alpha_2}_{i_2i_{2l}}\\ \vdots & \vdots & \ddots & \vdots\\ H^{\alpha_l}_{i_{2l-1}i_1} & H^{\alpha_l}_{i_{2l-1}i_2} & \dots & H^{\alpha_l}_{i_{2l-1}i_{2l}}\\ H^{\alpha_l}_{i_{2l}i_1} & H^{\alpha_l}_{i_{2l}i_2} & \dots & H^{\alpha_l}_{i_{2l}i_{2l}} \end{bmatrix}
\end{aligned}
\tag{9-4}
$$

9.4 The reason for expressing these polynomials as sums of $2l\times 2l$ determinants will become clear shortly. We also remark that if $h\in \mathrm{II}(V_0)$ and $H\in\mathrm{EII}(T)$ is the extension of h to $\mathrm{EII}(T)$, that is $H(u,v)=h(Pu,Pv)$ where $P:T\to V_0$ is the orthogonal projection, then it is easily checked that

$$w_{2l}(h)=w_{2l}(H) \tag{9-5}$$

For $h\in\mathrm{II}(V_0)$ (*resp.* $H\in\mathrm{EII}(T)$) and for each α with $p+1\le\alpha\le n$ (*resp.* $1\le\alpha\le n$) define a selfadjoint linear map $h^\alpha: V_0\to V_0$ (*resp.* $H^\alpha: T\to T$) by

$$
\begin{aligned}
\langle h^\alpha u, v\rangle &= \langle h(u,v), e_\alpha\rangle \quad \text{all} \quad u,v\in V_0,\ p+1\le\alpha\le n\\
\langle H^\alpha u, v\rangle &= \langle H(u,v), e_\alpha\rangle \quad \text{all} \quad u,v\in T,\ 1\le\alpha\le n
\end{aligned}
$$

In the case that h is the second fundamental form of a submanifold of a Riemannian manifold then h^α is just the usual Weingarten map of the submanifold corresponding to the normal direction e_α.

We now introduce a numerical invariant of $h\in\mathrm{II}(V_0)$ and $H\in\mathrm{EII}(T)$ which is closely related to the relative nullity of the second fundamental of a submanifold introduced by Chern and Kuiper in their paper [7] on the nonexistence of isometric imbeddings of low codimension.

9.5 Definition. *If $h\in\mathrm{II}(V_0)$ and $H\in\mathrm{EII}(T)$ then define the relative rank of h and H by*

$$
\begin{aligned}
\text{relative rank}\,(h) &= \dim\Big(\sum_{\alpha=p+1}^{n}\text{image}\,(h^a)\Big)\\
\text{relative rank}\,(H) &= \dim\Big(\sum_{\alpha=1}^{n}\text{image}\,(H^a)\Big).
\end{aligned}
$$

9.6 We will now explain the relation of the relative rank to Chern and Kuiper's relative nullity (in doing this we follow [14] Vol. II Note 16 on page 374) and explain what it means geometrically. If $h\in\mathrm{II}(V_0)$ then Chern and Kuiper introduce the subspace

$$
\begin{aligned}
\mathcal{N} &= \{u\in V_0 : h(u,v)=0 \quad \text{for all} \quad v\in V_0\}\\
&= \bigcap_{\alpha=p+1}^{n}\text{Kernel}(h^\alpha)
\end{aligned}
\tag{9-6}
$$

of V_0 and define

$$\text{relative nullity}\,(h) = \dim \mathcal{N}(h) \tag{9-7}$$

Because each h^α is selfadjoint

$$\text{Kernel}(h^\alpha) = V_0 \cap (\text{Image}(h^\alpha))^\perp$$

and therefore

$$\mathcal{N}(h) = \bigcap_{\alpha=p+1}^{n} \text{Kernel}(h^\alpha) = V_0 \cap \left(\sum_{\alpha=p+1}^{n} \text{Image}(h^\alpha) \right)^\perp . \tag{9-8}$$

It follows that the sum of the relative rank and the relative nullity of h is $p = \dim(V_0)$. Thus the two notations contain the same information about h. We note for future reference that analogous to the last equation there is a decomposition

$$\begin{aligned} \sum_{\alpha=1}^{n} \text{Image}(H^\alpha) &= \sum_{\alpha=1}^{n} \text{Kernel}(H^\alpha) \\ &= \left(\bigcap_{\alpha=1}^{n} \text{Kernel}(H^\alpha) \right)^\perp \\ &= \{u : H(u,v) = 0 \quad \text{for all} \quad v \in T^\perp\} \end{aligned} \tag{9-9}$$

To give some geometric meaning to the relative rank first define an isometric immersion $f : M \to \mathbb{R}^n$ from a p dimension Riemannian manifold M to be a rank k cylinder if and only if there is a k dimension Riemannian manifold M', an isometry $\phi : M \to M' \times \mathbb{R}^{p-k}$ (product metric) and an isometric immersion $f' : M' \to \mathbb{R}^{n-p+k}$ so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & \mathbb{R}^n \\ \phi \big\downarrow & & \big\downarrow \text{Id} \\ M' \times \mathbb{R}^{p-k} & \xrightarrow[f' \times \text{Id}]{} & \mathbb{R}^{n-p+k} \times \mathbb{R}^{p-k} \end{array}$$

commutes. If M is a rank k cylinder in $\mathbb{R}^n$ then the relative rank of the second fundamental form of M is at most k and relative nullity at least $p-k$ at all points of M and the relative rank will be exactly k at the "generic" point of the "generic" rank k cylinder. Conversely Hartmann [13] has shown that if M is a complete immersed submanifold of $\mathbb{R}^n$ with nonnegative sectional curvatures whose second fundamental form has relative rank k at each point then M is immersed as a rank k cylinder. Thus the relative rank of the second fundamental form of a submanifold of $\mathbb{R}^n$ is a measure of the number of independent directions in which M is curving in $\mathbb{R}^n$.

9.7 It is possible to compute the relative rank of $h \in \text{II}(V_0)$ (*resp.* $H \in \text{EII}(T)$) in a straight forward manner. Consider the components $[h^\alpha_{ij}]$ (*resp.* $[H^\alpha_{ij}]$) of h

(*resp.* H) as a $p(n-p)$ by p (*resp.* n^2 by n) matrix with rows indexed by the pairs (α, i) $1 \le i \le p$; $p+1 \le \alpha \le n$ (*resp.* $1 \le i, \alpha \le n$) and columns indexed by j with $1 \le j \le p$ (*resp.* $1 \le j \le n$). Then the relative rank of h (*resp.* H) is the same as the rank of the matrix $[h^\alpha_{ij}]$ (*resp.* $[H^\alpha_{ij}]$). To see this note that the components of h in the basis $e_1, \ldots, e_p$ are $(h^\alpha_{i1}, \ldots, h^\alpha_{ip})$. Thus the columns of $[h^\alpha_{ij}]$ span the space $\sum_{\alpha=p+1}^n \mathrm{Image}(h^\alpha)$ which shows the rank of $[h^\alpha_{ij}]$ is as claimed. The same argument proves the claim for $[H^\alpha_{ij}]$. The definition of the relative rank and the definitions of $w_{2l}(h)$ and $w_{2l}(H)$ as the sum of determinants of $2l$ by $2l$ submatrices of $[h^\alpha_{ij}]$ and $[H^\alpha_{ij}]$ implies the following;

9.8 Proposition. *The polynomial, w_{2l} on $\mathrm{II}(V_0)$ (resp. on $\mathrm{EII}(T)$) is invariant under $O(V_0) \times O(V_0^\perp)$ (resp. $O(T)$) and if $h \in \mathrm{II}(V_0)$ (resp. $H \in \mathrm{EII}(T)$) has relative rank less than $2l$ then*

$$w_{2l}(h) = 0, \qquad w_{2l}(H) = 0 \tag{9-10}$$

Our characterization of the polynomials is a converse of the last proposition.

9.9 Theorem. *Let $\mathcal{P}$ be a non-zero polynomial on $\mathrm{II}(V_0)$ such that*

(a) *$\mathcal{P}$ is homogeneous of degree k*
(b) *$\mathcal{P}$ is invariant under $O(V_0) \times O(V_0^\perp)$*
(c) *$\mathcal{P}(h) = 0$ for all $h \in \mathrm{II}(V_0)$ with*

$$\text{relative rank}(h) < k.$$

Then k is even, say $k = 2l$, and $\mathcal{P}$ is a constant multiple of w_{2l}.

9.10 Remark. The theorem implies, using the terminology of the introduction, if $\mathcal{P}$ is an invariant polynomial defined on the second fundamental forms of p dimensional submanifolds which is homogeneous of degree k and vanishes identically on the second fundamental forms of the cylinders of rank less than k then $k = 2l$ and $\mathcal{P} = cw_{2l}$ for some real number c. This gives a more or less geometric characterization of the polynomials w_{2l}.

9.11 Lemma. *Let $M_{m,n}$ be the space of m by n matrices and let $\mathcal{P}$ be a homogeneous polynomial of degree k on $M_{m,n}$ that vanishes on all elements of $M_{m,n}$ of rank less than k. Then $\mathcal{P}$ is a linear combination of the polynomials $D^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ for some $1 \le i_1 < \cdots i_k \le m$, $1 \le j_1 < \cdots < j_k \le n$ where*

$$D^{i_1 \ldots i_k}_{j_1 \ldots j_k}(X) = \det \begin{bmatrix} X_{i_1 j_1} & \ldots & X_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ X_{i_k j_1} & \ldots & X_{i_k j_k} \end{bmatrix}$$

and $X = [X_{ij}]$.

Proof. The polynomial $\mathcal{P}$ will be a linear combination of terms $(X_{i_1 j_1})^{a_1} \cdots (X_{i_l j_l})^{a_l}$ with $a_1 + \cdots + a_l = k$ and $a_t \ge 1$. We claim the coefficient of such a term is zero unless each $a_t = 1$ or, what is the same thing, unless $l = k$. To see this let $M(\lambda_1, \ldots, \lambda_k)$ be the matrix with λ_t in the (i_t, j_t)-th place and zero in all other entries. When $l < k$ this matrix always has rank less than k. Thus the polynomial that sends $(\lambda_1, \ldots, \lambda_l)$ to $\mathcal{P}(M(\lambda_1, \ldots, \lambda_l))$ vanishes identically. Therefore the coefficient of $(X_{i_1 j_1})^{a_1} \cdots (X_{i_l j_l})^{a_l}$ vanishes.

This implies $\mathcal{P}$ is a linear combination of terms of the form $X_{i_1 j_1} \cdots X_{i_k j_k}$. We now claim the coefficient of any such term is zero unless $i_1, \ldots, i_k$ are all distinct and also $j_1, \ldots, j_k$ are all distinct. This time let $M(\lambda_1, \ldots, \lambda_k)$ be the matrix with λ_t in the (i_t, j_t)-th place and all other entries zero. If $i_1, \ldots, i_k$ are not all distinct then this matrix will have at most $k-1$ nonzero rows and thus its rank is less than k. Therefore $\mathcal{P}(M(\lambda_1, \ldots, \lambda_k)) \equiv 0$ and so the coefficient of $X_{i_1 j_1} \cdots X_{i_k j_k}$ must vanish. A similar argument works in the case $j_1, \ldots, j_k$ are not distinct.

Therefore the nonzero terms of $\mathcal{P}$ are each of the form (constant)$X_{i_1 j_1} \cdots X_{i_k j_k}$ with $i_1 < \cdots < i_k$ and $j_1, \ldots, j_k$ distinct. It follows that $\mathcal{P}$ can be written as

$$\mathcal{P}(X) = \sum_{\substack{i_1 < \cdots < i_k \\ j_1 < \cdots < j_k}} \mathcal{P}^{i_1 \ldots i_k}_{j_1 \ldots j_k}(X) \tag{9-12}$$

where

$$\mathcal{P}^{i_1 \ldots i_k}_{j_1 \ldots j_k}(X) = \sum_{\sigma} C^{i_1 \ldots i_k}_{\sigma(j_1) \ldots \sigma(j_k)} X_{i_1 \sigma(j_1)} \cdots X_{i_k \sigma(j_k)}$$

and the sum is over all permutations of $j_1, \ldots, j_k$. For each $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ and each $k \times k$ matrix $A = [a_{st}]$ let $M(A)$ be the $m \times n$ matrix with a_{st} in the (i_s, j_t)-th place and zero in all other places. From the last two equations it follows $\mathcal{P}(M(A)) = \mathcal{P}^{i_1 \ldots i_k}_{j_1 \ldots j_k}(M(A))$ The rank of $M(A)$ is the same as the rank of A and therefore $P(M(A))$ vanishes if two rows of A are equal. Also, from equation (9-13), it is clear that $P(M(A))$ is a k linear function of the rows of A. But it is well known that the only functions on the k by k matrices that are k linear as a function of the rows and vanish whenever two rows are the same are the constant multiples of the determinant. Restated this implies that $\mathcal{P}^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ is a constant multiple of $D^{i_1 \ldots i_k}_{j_1 \ldots j_k}$. Using this in (9-12) implies the lemma.

9.12 We now return to the proof of theorem 9.9. For pairs $(\alpha_1, i_1), \ldots, (\alpha_k, i_k)$ and $j_1, \ldots, j_k$ define

$$D^{(\alpha_1, i_1) \ldots (\alpha_k, i_k)}_{j_1 \ldots j_k}(h) = \det \begin{bmatrix} h^{\alpha_1}_{i_1 j_1} & \ldots & h^{\alpha_1}_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ h^{\alpha_k}_{j_1} & \ldots & h^{\alpha_k}_{i_k j_k} \end{bmatrix} \tag{9-14}$$

Then the last lemma and the remarks in paragraph 9.7 imply

$$\mathcal{P}(h) = \sum_{\substack{(\alpha_1, i_1) < \cdots < (\alpha_k, i_k) \\ j_1 < \cdots < j_k}} C^{(\alpha_1, i_1) \ldots (\alpha_k, i_1)}_{j_1 \ldots j_k} D^{(\alpha_1, i_1) \ldots (\alpha_k, i_1)}_{j_1 \ldots j_k}(h) \tag{9-15}$$

where $(\alpha, i) < (\beta, j)$ iff $i < j$ or $i = j$ and $\alpha < \beta$. We now wish to find the $C^{(\alpha_1, i_1) \ldots (\alpha_k, i_1)}_{j_1 \ldots j_k}$'s by evaluating $\mathcal{P}$ on particular choices of the h's. In doing this it is much easier if we do not have to assume that h is symmetric. Therefore let $B(V_0)$ be the vector space of all bilinear maps from $V_0 \times V_0$ to V_0. If $h \in B(V_0)$

define the components h^α_{ij} of h by equation (8-7) and the relative rank of h to be the rank of the matrix $[h^\alpha_{ij}]$ where rows are indexed by pairs (α, i) and columns by j. Clearly $\mathrm{II}(V_0)$ is a subspace of $B(V_0)$. We now claim that $\mathcal{P}$ can be extended to a polynomial $\widehat{\mathcal{P}}$ on $B(V_0)$ That is:

(a) homogeneous of degree k on $B(V_0)$,
(b) vanishes on elements of $B(V_0)$ of relative rank less than k,
(c) $\widehat{\mathcal{P}}$ is invariant under the action of $O(V_0) \times O(V_0^\perp)$, and
(d) $\widehat{\mathcal{P}}(h) = \mathcal{P}(h)$ for all $h \in \mathrm{II}(V_0)$.

To see this note the polynomials $D^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k}$ given by (9-14) make sense as polynomials on $B(V_0)$. Therefore we can extend $\mathcal{P}$ to $B(V_0)$ by the formula (9-15) and the resulting polynomial on $B(V_0)$ will satisfy (a), (b), and (d). The group $O(V_0) \times O(V_0^\perp)$ is compact and its action on $B(V_0)$ preserves relative rank and leaves the subspace $\mathrm{II}(V_0)$ invariant. Therefore we can average over $O(V_0) \times O(V_0^\perp)$ to get a polynomial $\widehat{\widehat{\mathcal{P}}}$ on $B(V_0)$ that satisfies (a), (b), (c) and (d). We drop the "hat" over $\widehat{\widehat{\mathcal{P}}}$ and just write $\mathcal{P}$. Using the last lemma again we can assume that $\mathcal{P}$ is defined on $B(V_0)$ and is of the form given by (9-15).

We now show that $C^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k} = 0$ unless $\{i_1, \dots, i_k\} = \{j_1, \dots, j_k\}$. Toward this end fix $(\alpha_1, i_1), \dots, (\alpha_k, i_k)$ and $j_1, \dots, j_k$. For each $k \times k$ matrix $A = [a_{st}]$ let $h(A)$ be the element of $B(V_0)$ with components defined by $h(A)^{\alpha_s}_{i_s j_t} = a_{st}$ and all other components zero. Then in the expansion (9-15) all but one of the terms vanishes so that

$$\begin{aligned}\mathcal{P}(h(A)) &= C^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k} D^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k}(h(A)) \\ &= C^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k} \det(A).\end{aligned}$$

Now use the invariance of $\mathcal{P}$ under $O(V_0)$. Let ρ be the element of $O(V_0)$ such that $\rho e_j = \varepsilon_j e_j$ for $1 \le j \le p$ and $\varepsilon_j = \pm 1$ to be chosen later. Then using equation (8-7) it follows that if h^α_{ij} are the components of h then the components of ρh are

$$(\rho h)^\alpha_{ij} = \varepsilon_i \varepsilon_j h^\alpha_{ij}$$

Let I_k be the k by k identity matrix. If $\{i_1, \dots, i_k\} \ne \{j_1, \dots, j_k\}$ then there is a $j_s \in \{j_1, \dots, j_k\}$ with $j_s \notin \{i_1, \dots, i_k\}$. Let $\varepsilon_j = +1$ for $j \ne j_s$ and $\varepsilon_j = -1$. Then by the last equation

$$(\rho h)(I_k) = h(I'_k)$$

where I'_k is I_k with the s-th diagonal element replaced by -1 and all other entries unchanged. Then the last three equations and the invariance under $O(V_0)$ imply

$$C^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k} = \mathcal{P}(h(I'_k)) = \mathcal{P}(h(I_k)) = -C^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k}$$

and therefore this coefficient vanishes when $\{i_1, \dots, i_k\} \ne \{j_1, \dots, j_k\}$.

The invariance of $\mathcal{P}$ under $O(V_0)$ implies that for fixed $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_k\}$ the coefficients $C^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k}$ are symmetric functions of $\alpha_1, \dots, \alpha_k$ (any permutation of the vectors $e_{p+1}, \dots, e_n$ can be done by an orthogonal matrix). Using this along with the fact that $C^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{j_1\dots j_k}$ vanishes when $\{i_1,\dots,i_k\}\neq\{j_1,\dots,j_k\}$ implies that equation (9-15) can be rewritten as

$$\mathcal{P}(h)=\sum_{\substack{\alpha_1,\dots,\alpha_k\\ i_1<\dots<i_k}} C^{(\alpha_1,i_1)\dots(\alpha_k,i_1)}_{i_1\dots i_k} D^{(\alpha_1,i_1)\dots(\alpha_k,i_1)}_{i_1\dots i_k}(h) \tag{9-16}$$

Fix $\beta_1,\dots,\beta_k$ and for each $j_1<\dots<j_k$ let $h(j_1,\dots,j_k)$ be the element of $B(V_0)$ with components $h(j_1,\dots,j_k)^{\beta_s}_{j_sj_s}=1$ for $1\le s\le k$ and all other components zero. Then in the expansion (9-16) for $\mathcal{P}(h(j_1,\dots,j_k))$ all but one term is zero and

$$D^{(\beta_1,j_1)\dots(\beta_k,j_k)}_{j_1\dots j_k}(h(j_1,\dots,j_s))=1.$$

Therefore

$$\mathcal{P}(h(j_1,\dots,j_k))=C^{(\beta_1,j_1)\dots(\beta_k,j_k)}_{j_1\dots j_k}$$

Given any other set $1\le j_1'<\dots<j_k'\le p$ then there is an element ρ of $O(V_0)$ with

$$\rho h(j_1,\dots,j_k)=h(j_1',\dots,j_k')$$

this is because every permutation of the vectors $e_1,\dots,e_p$ can be realized by an element of $O(V_0)$. The last two equations and the invariance of $\mathcal{P}$ under $O(V_0)$ implies $C^{(\beta_1,j_1)\dots(\beta_k,j_k)}_{j_1\dots j_k}=C^{(\beta_1,j_1')\dots(\beta_k,j_k')}_{j_1'\dots j_k'}$ and thus these coefficients are independent of $j_1,\dots,j_k$. Therefore equation (9-16) can be rewritten

$$\begin{aligned}\mathcal{P}(h)&=\sum_{\substack{i_1<\dots<i_k\\ \alpha_1,\dots,\alpha_k}} C_{\alpha_1,\dots,\alpha_k} D^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{i_1\dots i_k}(h)\\ &=\frac{1}{k!}\sum_{\substack{i_1,\dots,i_k\\ \alpha_1,\dots,\alpha_k}} C_{\alpha_1,\dots,\alpha_k} D^{(\alpha_1,i_1)\dots(\alpha_k,i_k)}_{i_1\dots i_k}(h)\\ &=\frac{1}{k!}\sum_{\substack{j_1,\dots j_k\\ i_1,\dots,i_k\\ \alpha_1,\dots,\alpha_k}} \delta^{i_1\dots i_k}_{j_1\dots j_k} h^{\alpha_1}_{i_1j_1}h^{\alpha_2}_{i_2j_2}\cdots h^{\alpha_k}_{i_kj_k}\end{aligned} \tag{9-17}$$

For each i, j with $1\le i,j\le p$ define a vector $h_{ij}\in V_0^{\perp}$ by $h_{ij}=h(e_i,e_j)$. Then $\mathcal{P}(h)$ can be viewed as a polynomial in the components of the p^2 vectors h_{ij}. The invariance of $\mathcal{P}$ under $O(V_0)$ implies, by the first main theorem on vector invariants for the orthogonal group (see [22] page 53), that $\mathcal{P}$ is a polynomial in the inner products

$$\langle h_{ij},h_{st}\rangle=\sum_{\alpha=p+1}^{n} h^{\alpha}_{ij}h^{\alpha}_{st}$$

Whence the degree of $\mathcal{P}$ must be even, say $k=2l$, and in the sum in (9-17) the upper indices in $h^{\alpha_1}_{i_1j_1}\cdots h^{\alpha_k}_{i_kj_k}$ must be contracted in pairs. This last fact implies that $C_{\alpha_1,\dots,\alpha_k}$ is independent of $\alpha_1,\dots,\alpha_k$ therefore (9-17) becomes

$$\mathcal{P}(h)=(\text{Constant})\sum_{\substack{i_1,\dots,i_{2l}\\ j_1,\dots,j_{2l}\\ \alpha_1,\dots,\alpha_l}} \delta^{i_1,\dots,i_{2l}}_{j_1,\dots,j_{2l}}\prod_{t=1}^{l}(h^{\alpha_t}_{i_{2t-1}j_{2t-1}}h^{\alpha_t}_{i_{2t}j_{2t}}).$$

This completes the proof of the theorem

9.13 Before going on to the proofs of the Weyl tube formula and the Chern-Federer kinematic formula in the next section we give two lemmas, the first of which will be used in the tube formula and the second used in the kinematic formula.

9.14 LEMMA. *If A and B are $l \times l$ matrices and the rank of A is less than k then the coefficient of λ^j in $\det(\lambda A + B)$ vanishes for $j \geq k$.*

9.15 LEMMA. *Let H_1, $H_2 \in \mathrm{EII}(T)$ and assume that the relative rank of H_1 is less than $2k$ Then for $l > k$ the coefficient of λ^j in $w_{2l}(\lambda H_1 + H_2)$ vanishes for $j \geq 2k$.*

9.16 PROOFS. In 9.14 first assume $\det(B) \neq 0$. Then $\det(\lambda A+B) = \det(B)\det(\lambda B^{-1}A+I)$. The matrix $B^{-1}A$ has rank less than k and therefore at least $l-k+1$ of the eigenvalues of $B^{-1}A$ are zero. Whence the result follow from the Cayley-Hamilton theorem. The restriction $\det(B) \neq 0$ is removed by a straight forward continuity argument.

To prove 9.15 use the form of the formula for $w_{2l}(\lambda H_1 + H_2)$ that expresses it as a linear combination of determinants of $2l \times 2l$ matrices. Then the remarks in paragraph 9.7 show that 9.15 reduces to 9.14.

10. The Weyl tube formula and the Chern-Federer kinematic formula.

10.1 In this section we return to the notation of paragraph 8.1, that is G/K is the n dimensional simply connected manifold of constant curvature c and G is the full isometry group of G/K. For $0 \leq 2l \leq n$ let w_{2l} be the polynomial defined on $\mathrm{EII}(T(G/K)_{\mathbf{o}})$ in definition 9.3. Then for each compact submanifold M of G/K (possibly with boundary) define $\mu_{2l}(M)$ to be the integral invariant

$$\begin{aligned} \mu_{2l}(M) &= I^{w_{2l}}(M) \\ &= \int_M w_{2l}(H_x^M)\,\Omega_M(x) \\ &= \int_M w_{2l}(h_x^M)\,\Omega_M(x) \qquad \text{(10-1)} \end{aligned}$$

where H_x^M is the extended second fundamental form, h_x^M is the second fundamental form of M at x and the equality between the second and third lines follows from equation (9-5). This shows that μ_{2l} can be considered an integral invariant in the sense of paragraph 4.6 (defined in terms of the second fundamental form of a submanifold on which G is transitive on the set of tangent spaces) or in the sense of paragraph 4.9 (defined in terms of the extended second fundamental form).

The invariants μ_{2l} were introduced by Hermann Weyl in his famous paper [21] on the volume of tubes in Euclidean space. To be specific let M be a closed compact imbedded p dimensional submanifold of $\mathbb{R}^n$ and let $\tau_r M$ be the tube of radius r about M, that is $\tau_r M$ is the set of points at a distance at most r from M. Then Weyl proved that for small r

$$\mathrm{Vol}(\tau_r M) = \sum_{0 \leq 2l \leq p} \gamma(n,p,l)\mu_{2l}(M) r^{n-p+2l}. \qquad \text{(10-2)}$$

where the $\gamma(n,p,l)$'s are constants only depending on the indicated numbers and which were given explicitly by Weyl.

10.2 We sketch a proof of Weyl's formula based on the characterization of the μ_{2l}'s given in the last section. Locally along M choose an orthonormal moving frame so that $e_1,\dots,e_p$ span the tangent space to M and $e_{p+1},\dots,e_n$ span the normal space to M. Let h^M be the second fundamental form of M in $\mathbb{R}^n$ and for $p+1\le\alpha\le n$ let $(h^M_x)^\alpha$ be the linear map on the tangent space to M at x given by $\langle (h^M_x)^\alpha X, Y\rangle = \langle h^M_x(X,Y), e_\alpha\rangle$ for all X, $Y\in TM_x$. Then a calculation, which Weyl informs us is "hardly ... more than what could have been accomplished by any student in a course of calculus", shows that

$$\mathrm{Vol}(\tau_r M) = \int_M \mathcal{P}(h^M_x)\,\Omega_M(x) \tag{10-3}$$

where $\mathcal{P}$ is the polynomial defined on $\mathrm{II}(TM_X)$ (the symmetric bilinear maps from $TM_x\times TM_x$ to $T^\perp M_x$) by

$$\mathcal{P}(h) = \int_{t^2_{p+1}+\cdots+t^1_n\le r^2} \det\left(I+\sum_{\alpha=p+1}^n t_\alpha h^\alpha\right) dt^\alpha\cdots dt^n \tag{10-4}$$

where I is the identity map on TM_x. (See the formula for $V(a)$ on page 464 of [21]). Define new polynomials $\mathcal{P}_0,\dots,\mathcal{P}_p$ on $\mathrm{II}(TM_x)$ by

$$\int_{x^2_{p+1}+\cdots+x^2_n\le 1} \det\left(I+\lambda\sum_{\alpha=p+1}^n x_\alpha h^\alpha\right) dx_{p+1}\cdots dx_n = \sum_{j=0}^p \lambda^j\mathcal{P}_j(h) \tag{10-5}$$

Then each $\mathcal{P}_j$ is homogeneous of degree j. If $A=[a_{\alpha\beta}]$, $p+1\le\alpha,\beta\le n$ is an $n-p$ by $n-p$ orthogonal matrix then a change of variable in the integral on the left of (10-5) shows that this integral, and thus the polynomials $\mathcal{P}_j$, are unchanged by replacing each h^α by $\sum_\beta a_{\alpha\beta}h^\beta$. This shows that $\mathcal{P}_j$ is invariant under $O(T^\perp M_x)$. Elementary properties of the determinant show that if $\rho\in O(TM_x)$ then replacing each h^α by $\rho h^\alpha\rho^{-1}$ in (l0-5) leaves the left side of (l0-5) unchanged. This shows that $\mathcal{P}_j$ is also invariant under $O(TM_x)$. Lemma 9.14 and the definition of relative rank implies that $\mathcal{P}_j(h)=0$ if the relative rank of h is less than j. Therefore theorem 9.9 implies $\mathcal{P}_j=0$ if j is odd and if $j=2l$ is even that

$$\mathcal{P}_{2l} = \gamma(n,p,l)w_{2l}$$

for some constant $\gamma(n,p,l)$. A change of variable in (10-4) implies

$$\mathcal{P}(h) = \sum_{0\le 2l\le p} \mathcal{P}_{2l}(h)r^{n-p+2l}$$

Using these last two equations in (10-3) proves Weyl's formula (10-2). The constants $\gamma(n,p,l)$ can be computed by letting $h_{p+1}=I$, $h_{p+2}=\cdots=h_n$ in (10-4).

Recall that $w_{2l}(h^M_x)$ can be expressed in terms of the $R^{st}_{ij}(h^M_x)$ which are the components of the curvature tensor of M. Therefore $\mathrm{Vol}(\tau_r M)$ is an intrinsic invariant of M and thus it is the same for all isometric imbeddings of M into $\mathbb{R}^n$. As is well known, this remarkable fact was used by Allendoerfer and Weil [1] to give the first proof of the generalized Gauss-Bonnet theorem which says that if M is a compact oriented Riemannian manifold of even dimension $2l$ then

$$\mu_{2l}(M) = (\text{Constant})\chi(M) \tag{10-6}$$

where $\chi(M)$ is the Euler characteristic of M. Their proof has been updated by Griffiths ([12] page 509 paragraph (iv)) gives a proof of (10-6) that has a lot of geometric appeal.

10.3 We now turn to the kinematic formula of Chern and Federer. Let M be a compact p dimensional and N a compact q dimensional submanifold of G/K (possibly with boundary). Assume that $0 \le 2l \le p+q-n$. Then Federer [8] and Chern [6] proved that

$$\int_G \mu_{2l}(M \cap gN)\,\Omega_G(g) = \sum_{0\le k\le l} C(n,p,q,k,l)\mu_{2k}(M)\mu_{2(l-k)}(N) \tag{10-7}$$

where each constant $c(n,p,q,k,l)$ only depends on the indicated parameters. In particular the $c(n,p,q,k,l)$ are independent the curvature of G/K. These constants were first computed in Chern's paper. Later Nijenhuis [16] found a much more compact expression for $c(n,p,q,k,l)$ and the reader is refereed to his paper for their value. Note that here and in Federer's paper we integrate over the full isometry group of G/K while Chern only integrates over the group of orientation preserving isometries. Thus the values of $c(n,p,q,k,l)$ used in (10-7) or in [8] will be twice as large as those in [6] and [16].

10.4 Actually Chern and Federer only proved (10-7) in the case where $G/K = \mathbb{R}^n$, the Euclidean space of n dimensions. But by the transfer principle it follows that (10-7) holds in all spaces of constant sectional curvature.

However it is of interest to give a proof that works in all cases. We now give such a proof based on the kinematic formula 7.2 and the algebraic result 9.9. By theorem 7.2 it is enough to prove (using $T = T(G/K)_{\mathbf{o}}$ and the identification of K with $O(T)$ given in paragraph 8.1) that

$$I^{w_{2l}}_{O(T)}(V_0,h_1,W_0,h_2) = \sum_{0\le k\le l} c(n,p,q,k,l)w_{2k}(h_1)w_{2(l-k)}(h_2)$$

for all $h_1 \in \mathrm{II}(V_0)$ and $h_2 \in \mathrm{II}(W_0)$. Define polynomials $\mathcal{P}_j$ on $\mathrm{II}(V_0)\bigoplus \mathrm{II}(W_0)$ by

$$\sum_{j=0}^{2l} \lambda^j \mathcal{P}_j(h_1,h_2) = I^{w_{2l}}_{O(T)}(V_0,\lambda h_1,W_0,h_2)$$

$$= \int_{O(T)} w_{2l}\left(\lambda G_{b^{-1}W_0}(V_0,h_1) + G_{V_0}(b^{-1},b^{-1}h_2)\right)\sigma(V_0^\perp, b^{-1}W_0^\perp)\,\Omega_{O(T)} \tag{10-9}$$

where the equality $G_{b^{-1}W_0}(V_0,\lambda h_1) = \lambda G_{b^{-1}W_0}(V_0,h_1)$ has been used.

The polynomial $\mathcal{P}_j(h_1,h_2)$ is homogeneous of degree j in h_1 and homogeneous of degree $2l-j$ in h_2. By the invariance properties of $I^{w_{2l}}_{O(T)}$ given in lemma 6.4

part (3) it follows that for fixed h_1 the polynomial $h_2 \mapsto \mathcal{P}_j(h_1, h_2)$ is invariant under $O(W_0) \times O(W_0^\perp)$ and for fixed h_2 that $h_1 \mapsto \mathcal{P}_j(h_1, h_2)$ is invariant under $O(V_0) \times O(V_0^\perp)$.

We now show that the relative rank $G_{b^{-1}W_0}(V_0, h_1)$ is less than or equal to the relative rank of h_1. Let r be the relative rank of h_1. Then by the results in paragraph 9.6 it follows

$$p - r = \dim\{u \in V_0 : h_1(u, v) = 0 \quad \text{for all} \quad v \in V_0\} \tag{10-10}$$

and by equation (9-9) we wish to show

$$\begin{aligned} n - r &\geq \dim\{u \in T : G_{b_*^{-1}W_0}(V_0, h_1)(u, v) = 0 \quad \text{for all} \quad v \in T\} \\ &= \dim\{u \in T : P^{V_0}_{b_*^{-1}W_0} h_1(Pu, Pv) = 0 \quad \text{for all} \quad v \in T\} \end{aligned} \tag{10-11}$$

But, as P is the orthogonal projection onto $b^{-1}W_0 \cap V_0 \subseteq V_0$, $Pu = 0 = Pv$ for all $u, v \in V^\perp$. Therefore to deduce (10-11) from (10-10) it is enough to prove

$$\begin{aligned} &\dim\{u \in V_0 : P^{V_0}_{b_*^{-1}W_0} h_1(Pu, Pv) = 0 \quad \text{for all} \quad v \in V_0\} \\ &\geq \dim\{u \in V_0 : h_1(u, v) = 0 \quad \text{for all} \quad v \in V_0\}. \end{aligned}$$

But this relation is elementary.

Fix an element $h_2 \in \mathrm{II}(W_0)$ and let $h_1 \in \mathrm{II}(V_0)$ have relative rank less than j. Then $G_{b^{-1}W_0}(V_0, h_1)$ also has relative rank less than j therefore, using lemma 9.15 in equation (10-9), it follows $\mathcal{P}_j(h_1, h_2) = 0$. Whence all the hypothesis of theorem 9.9 have been verified for the polynomial $h_1 \mapsto \mathcal{P}_j(h_1, h_2)$. Thus 9.9 implies that j is even, say $j = 2k$, and that

$$\mathcal{P}_j(h_1, h_2) = \mathcal{P}_{2k}(h_1, h_2) = C(n, p, q, k, l, h_2) w_{2k}(h_1). \tag{10-12}$$

But by easy variants of the arguments just used we see that $h_2 \mapsto C(n, p, q, k, l, h_2)$ is a polynomial on $\mathrm{II}(W_0)$ which is invariant under $O(W_0) \times O(W_0^\perp)$, homogeneous of degree $2l - 2k$ that vanishes on elements of $\mathrm{II}(W_0)$ of relative rank less than $2l - 2k$. Therefore another application of theorem 9.9 implies that

$$C(n, p, q, k, l, h_2) = C(n, p, q, k, l) w_{2l-2k}(h_2). \tag{10-13}$$

Using (10-13) in (10-12) and the result of that in (10-9) implies the required formula (10-8). This completes the proof of the Chern-Federer formula (10-7). To find the constants $C(n, p, q, k, l)$ let M be the p dimensional unit sphere imbedded in $\mathbb{R}^n$ in the usual way, N the q dimensional sphere of radius a in (10-7) and evaluate both sides directly. This is a nontrivial calculation and the reader is referred to the papers of Chern and Nijenhuis cited above.

Appendix: Fibre integrals and the smooth coarea formula.

In this appendix we give a proof of Federer's coarea formula for smooth maps between Riemannian manifolds that avoids the measure theoretic machinery needed in the case of Lipschitz maps. For the proof in the more general case the reader is referred to the paper [8] of Federer or to his book [9].

Let M^{n+m} be a smooth Riemannian manifold of dimension $n+m$, N^n a smooth Riemannian manifold of dimension n (we allow the possibility that $m=0$) and $f: M^{m+n} \to N^n$ a smooth map. Recall that $x \in M$ is a **regular point** of f if and only if $f_{*x}: TM_x \to TN_{f(x)}$ is surjective and a critical point otherwise. A point y in N is a regular value of f if and only if every point of $f^{-1}[y]$ is a regular point of f (and by convention y is a regular value if $f^{-1}[y]$ is empty) and is a critical value if it is not a regular value. We are guaranteed the existence of regular values by:

Sard's Theorem. *The set of critical values of f has measure zero.*

If y is a regular value of f then, by the implicit function theorem, $f^{-1}[y]$ is either empty or a closed imbedded m dimensional submanifold of M. Therefore if h is a smooth function on M with compact support the function $y \mapsto \int_{f^{-1}[y]} h\,\Omega_{f^{-1}[y]}$ (set this integral to be zero when $f^{-1}[y]$ is empty) is defined for all regular values of f and thus almost everywhere on N. The coarea formula gives the integral of this function in terms of an integral over M involving the Jacobian Jf of f which we now define.

$$Jf(x) = \begin{cases} 0 & \text{if } x \text{ is a critical point of } f, \\ \|f_*e_1 \wedge \cdots \wedge f_*e_n\| & \begin{cases} \text{if } x \text{ is a regular value of } f \text{ and } e_1, \dots, e_n \\ \text{is an orthonormal basis of Kernel}(f_{*x})^\perp. \end{cases} \end{cases}$$

Thus $Jf(x) \neq 0$ if and only if x is a regular point of f. If x is a regular point of f and $e_1, \dots, e_n$ is an orthonormal basis of Kernel$(f_{*x})^\perp$ then it can be verified that $Jf(x)$ is also given by

$$Jf(x) = |\Omega_N(f_*e_1, \dots, f_*e_n)| \tag{A-1}$$

where Ω_N is the volume form on N.

The coarea formula is

$$\int_N \int_{f^{-1}[y]} h\,\Omega_{f^{-1}[y]}\,\Omega_N(y) = \int_M h(x) Jf(x)\,\Omega_M(x) \tag{A-2}$$

where h is any Borel measurable function defined almost everywhere on M so that the integral on the right is finite. We will prove this formula in the case h is smooth with compact support, the general case then follows by a standard approximation argument.

To prove (A-2) we first note, by use of a partition of unity, that the problem is local and thus we can assume that both M and N are oriented. Let $M^* = \{x : Jf(x) \neq 0\}$ be the set of regular points of f. Then clearly

$$\int_M h(x) Jf(x)\, \Omega_M(x) = \int_{M^*} h(x) Jf(x)\, \Omega_{M^*}(x).$$

By definition y is a regular value of f if and only if $f^{-1}[y] = f^{-1}[y] \cap M^*$, and by Sard's theorem this is true for almost all $y \in N$. Whence

$$\int_N \int_{f^{-1}[y]} h\, \Omega_{f^{-1}[y]}\, \Omega_N(y) = \int_N \int_{f^{-1}[y] \cap M^*} h\, \Omega_{f^{-1}[y] \cap N^*}\, \Omega_N(y).$$

The last two equations show that in proving (A-2) we can replace M by M^* and thus assume that every point of M is a regular point of f, and that $f^{-1}[y]$ is an m dimensional submanifold of M for all $y \in N$ for which $f^{-1}[y]$ is not empty.

For each $x \in M$ it is possible to choose an oriented orthonormal basis $e_1, \dots, e_{n+m}$ of TM_x in such a way that $e_{m+1}, \dots, e_{n+m}$ is an orthonormal basis of $\mathrm{Kernel}(f_{*x})^\perp = T(f^{-1}[y])^\perp$ (where $y = f(x)$) such that $f_* e_{m+1}, \dots, f_* e_{n+m}$ is an oriented basis of TN_y. Then give the submanifold $f^{-1}[y]$ the orientation such that $e_1, \dots, e_m$ is an oriented basis of $T(f^{-1}[y])_x$. Let $\sigma_1, \dots, \sigma_{n+m}$ be the one forms dual to $e_1, \dots, e_{n+m}$ and define an m form ω_1 and an n form ω_2 on M by

$$\omega_1 = \sigma_1 \wedge \cdots \wedge \sigma_m$$
$$\omega_2 = \sigma_{m+1} \wedge \cdots \wedge \sigma_{n+m}$$

The forms ω_1 and ω_2 and the orientation on $f^{-1}[y]$ are defined independently of the choice of the orthonormal basis $e_1, \dots, e_{n+m}$. Also

$$\omega_1 \wedge \omega_2 = \Omega_M$$
$$\omega_1|_{f^{-1}[y]} = \Omega_{f^{-1}[y]} \quad \text{for all} \quad y \in N \tag{A-3}$$

From (A-1)

$$f^* \Omega_N(e_{m+1}, \dots, e_{n+m}) = \Omega_N(f_* e_{m+1}, \dots, f_* e_{n+m}) = Jf(x)$$

and as $e_1, \dots, e_m \in \mathrm{Kernel}(f_*)$,

$$f^* \Omega_N(e_{i_1}, \dots, e_{i_n}) = \Omega_N(f_* e_{i_1}, \dots, f_* e_{i_n}) = 0 \quad \text{if some } i_j \leq m$$

The last two equations imply $f^* \Omega_N = (Jf)\omega_2$. This, along with (A-3), give an infinitesimal version of the coarea formula; if $x \in M$, $y \in f(x)$ then

$$\Omega_{f^{-1}[y]} \wedge f^* \Omega_M = \omega_1 \wedge Jf(x) \omega_2 = Jf(x)\, \Omega_M.$$

Using this formula and $\omega_1|_{f^{-1}[y]} = \Omega_{f^{-1}[y]}$ proving the coarea formula reduces to showing

$$\int_N \int_{f^{-1}[y]} h\, \omega_1\, \Omega_N(y) = \int_M h(x)\, \omega_1 \wedge f^* \Omega_N.$$

This is implied by the following elementary and well known

LEMMA ON FIBER INTEGRATION. *Let $f : M^{n+m} \to N^n$ be a submersion of smooth oriented manifolds, α an m form on M and β an n form on N. Then for any smooth compactly supported function h on M*

$$\int_N \left(\int_{f^{-1}[y]} h\alpha \right) \beta(y) = \int_M h\, \alpha \wedge f^* \beta$$

The proof of this is straight forward, by the implicit function theorem and a partition of unity we may assume $M = \mathbb{R}^{n+m}$, that N is $\mathbb{R}^n$ imbedded into M as the first n coordinates and that f is the projection onto the first n coordinates. The lemma then just reduces to Fubini's theorem. Details are left to the reader.

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