CHARACTERIZATION OF TANTRIX CURVES

RALPH HOWARD
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, S.C. 29208, USA
HOWARD@MATH.SC.EDU

Contents

1. Introduction

Let S^1 be the unit circle realized as $\mathbf{R}/\mathbf{Z}$. Then a ***regular closed curve*** in $\mathbf{R}^n$ is a smooth mapping $c\colon S^1 \to \mathbf{R}^n$ so that the velocity vector $c'(t)$ never vanishes. The ***tantrix*** of such a curve is the map $\mathbf{t}\colon S^1 \to S^{n-1}$ given by $\mathbf{t}(t) = c'(t)/\|c'(t)\|$, that is the unit tangent to c parallel translated to the origin. Our goal is to understand which curves $\mathbf{t}\colon S^1 \to S^{n-1}$ can be realized as the tantrix to a regular closed curve. A first step is the following, due to Löwner, is a necessary condition for a curve to be a tantrix. This is attributed to Löwner by both Fenchel [1, p. 39] and Pólya and Szegő [2, Band II S. 165 und 391 Aufgabe, 13.]. I have not been able to track down the original paper of Löwner.

Theorem 1.1 (Löwner)**.** *If a curve $\mathbf{t}\colon S^1 \to S^{n-1}$ is a tantrix then the origin is in the convex hull of the image of $\mathbf{t}$. This implies that every totally geodesic S^{n-2} in S^{n-1} meets $\mathbf{t}$ in at least one point.*

Proof. Let L be the length of c. Then if $\gamma(s)$ is the unit speed parameterization of c we have that γ is defined on $\mathbf{R}/L\mathbf{Z}$ and that $\gamma'(s)$ is a reparameterization of the tantrix $\mathbf{t}$ of c. Therefore $\mathbf{t}$ and γ' have the same image. But the center of mass of γ' is the origin:

$$\int_0^L \gamma'(s)\,ds = \gamma(L) - \gamma(0) = 0.$$

This clearly implies that 0 is in the convex hull of the image of γ' with is the same as the image of $\mathbf{t}$.

Date: May 1999.

Also for any nonzero vector $a \in \mathbf{R}^n$ we have $a \cdot \int \int_0^L \gamma'(s)\,ds = \int_0^L a \cdot \gamma'(s)\,ds = 0$. (Here $\cdot$ is the standard inner product on $\mathbf{R}^n$.) Any totally geodesic S^{n-2} in S^{n-1} is of the form $S^{n-1} \cap a^\perp$ for a nonzero vector a. Therefore as the image of γ' is connected the equality $\int_0^L a \cdot \gamma'(s)\,ds = 0$ implies that either $a \cdot \gamma'(s) \equiv 0$ or that $a \cdot \gamma'(s)$ changes signs. In either there are points of γ' on $a^\perp \cap S^{n-1}$. Thus $\mathbf{t}$ meets every totally geodesic S^{n-2} in at least one point. □

This necessary condition comes close to being sufficient. The following is, in the proper circles, a well known folk theorem (I learned of the result and its proof from Mike Gage), however the only explicit reference I have seen is the original paper of Fenchel on the total curvature of space curves [1] where a not quite correct version is stated and attributed to Löwner. The reference Fenchel gives is Pólya and Szegő [2, Band II S. 165 und 391 Aufgabe, 13.], but in fact Pólya and Szegő state and prove a version of Theorem 1.1, which they also credit to Löwner, but do not state or prove the converse.

Theorem 1.2. *Let $\mathbf{t}\colon S^1 \to S^n$ so that the image of $\mathbf{t}$ is not contained in any totally geodesic S^{n-2}. Then $\mathbf{t}$ is the tantrix of a regular space curve if and only if 0 is in the interior of the convex hull of the image of $\mathbf{t}$. Explicitly if $\mathbf{t}$ is of continuous and 0 is in the interior of the convex hull of the image of $\mathbf{t}$, then there is a C^1 curve $c\colon S^1 \to \mathbf{R}^n$ so the that $\mathbf{t}(t) = c'(t)/\|c'(t)\|$. (If $\mathbf{t}$ is C^k then c is C^{k+1}.)*

Remark 1.3. It is all right if $\mathbf{t}$ is not injective, or if it is not an immersion. □

Remark 1.4. It is sometimes stated that a sufficient condition for a curve in S^2 to be a tantrix is that it meet every great circle. However this only implies that the origin is in the convex hull of the image of the curve, not that it is in the interior of the of the convex hull. For an explicit example

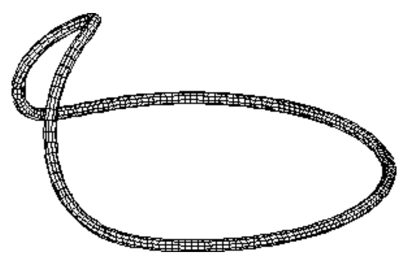

FIGURE 1. A curve on the sphere S^2 that meets every great circle in at least two points, but which is not the tantrix of any smooth curve.

consider the curve of Figure (1). This curve agrees with a great circle for interval large enough to contain a pair of antipodal points of the great circle formed by intersecting the sphere with the x-y plane, and the curve is in the upper half space $\{z > 0\}$. The origin is in the convex hull of this curve, and so every great circle will meet this curve in at least two points. However from the proof of Theorem 1.1 we have that if γ is a unit speed curve with the pictured curve as tantrix, then the center of mass of γ' is 0. However any parameterization of the pictured curve must have center of mass above the x-y plane. So it is impossible that the curve is a tantrix of any C^1 curve.

A condition equivalent to $\mathbf{t}$ having 0 in its convex hull is that $\mathbf{t}$ is not contained in closed hemisphere if S^{n-1}. $\square$

2. Proof of Theorem 1.2

First assume that $\mathbf{t}$ is a tantrix of some curve, and that $\mathbf{t}$ is not contained in any totally geodesic S^{n-2}. Then, just as in the proof of Theorem 1.1, there is a unit speed curve γ of length L so that γ' is a reparameterization of $\mathbf{t}$ and the center of mass of γ' is the origin. If 0 is not an interior point of the convex hull of $\mathbf{t}$ then it is on the boundary of the convex hull. Therefore by standard results from convexity there is a supporting hypersurface to the convex hull at the origin. This means that there is a unit vector a so that $a\cdot\gamma'(s) \geq 0$ for all s. But then $\int_0^L a\cdot\gamma'(s)\,ds = 0$ and continuity implies that $a\cdot\gamma'(s) \equiv 0$, and thus γ', and therefore also $\mathbf{t}$, is contained in the totally geodesic S^{n-2} $a^\perp S^{n-1}$. This contradicts a hypothesis and so we must have that 0 is in the interior of the convex hull of $\mathbf{t}$.

Now assume that 0 is in the interior of convex hull of $\mathbf{t}$. We are viewing S^1 as $\mathbf{R}/\mathbf{Z}$ so that $\mathbf{t}$ can be thought of as a map $\mathbf{t}\colon \mathbf{R} \to S^2$ with $\mathbf{t}(t+1) = \mathbf{t}(t)$. We will try to find a positive function $v\colon \mathbf{R} \to (0,\infty)$ with $v(t+1) = v(t)$ and so that the center of mass of the product $v\mathbf{t}$ is the origin. That is

$$\int_0^1 v(\tau)\mathbf{t}(\tau)\,d\tau = 0, \tag{2.1}$$

Then periodicity implies that for all t

$$\int_t^{t+1} v(\tau)\mathbf{t}(\tau)\,d\tau = 0. \tag{2.2}$$

Set

$$c(t) := \int_0^t v(\tau)\mathbf{t}(\tau)\,d\tau.$$

The integral condition (2.2) implies $c(t+1) = c(t)$ so c is a closed curve $c\colon S^1 \to \mathbf{R}^n$. Also $c'(t) = v(t)\mathbf{t}(t)$ and as v is positive this implies that c is an immersion. Finally $c'(t) = v(t)\mathbf{t}(t)$ clearly implies that $\mathbf{t}$ is the tantrix of c.

We now show the existence of the function v so that (2.1) holds. As the origin is in the interior of the convex hull of c we can use a theorem of Carathéodory's Theorem[1] (cf. [3, p. 3]) to find distinct $t_0,\dots,t_n \in [0,1)$ and $\alpha_0,\dots,\alpha_n > 0$ so that

$$\sum_{i=0}^n \alpha_i \mathbf{t}(t_i) = 0.$$

[1] This theorem is that if $A \subseteq \mathbf{R}^n$, then any point of the convex hull of A is a convex combination of $\leq n+1$ points of A.

By continuity there is a $\delta > 0$ so that if $P_0, \dots, P_n \in \mathbf{R}^n$ then

$$\|c(t_i) - P_i\| < \delta \text{ for } i = 0, \dots, n \implies 0 \in \text{convex hull}\,\{P_0, \dots, P_n\}. \tag{2.3}$$

Now for $i = 0, \dots, n$ there is a smooth positive C^∞ function v_i on $\mathbf{R}$ with $v_i(t+1) = v_i(t)$, $\int_0^1 v_i(t)\,dt = 1$ and that approximates the point mass at t_i well enough that

$$\left\| c(t_i) - \int_0^1 v_i(t)\mathbf{t}(t)\,dt \right\| < \delta.$$

Therefore the implication (2.3) and Carathéodory's Theorem yields $\beta_0, \dots, \beta_n \geq 0$ with $\sum_{i=0}^n \beta_i = 1$ and

$$0 = \sum_{i=0}^n \beta_i \int_0^1 v_i(t)\mathbf{t}(t)\,dt = \int_0^1 \sum_{i=0}^n \beta_i v_i(t)\mathbf{t}(t)\,dt = \int_0^1 v(t)\mathbf{t}(t)\,dt$$

where $v(t) = \sum_{i=0}^n \beta_i v_i(t)$. This gives us the v so that the desired relation (2.1) holds and completes the proof.

3. The Case of Curves Symmetric with respect to a Group.

We can view $S^1 = \mathbf{R}/\mathbf{Z}$ as a group in the usual way. Let G be a closed subgroup of S^1. Then either G is a finite cyclic group, or $G = S^1$. Given such a subgroup we a map $\gamma\colon S^1 \to \mathbf{R}^n$ has a G ***symmetry*** iff there is a continuous group homomorphism $\rho\colon G \to O(n)$ so that for all $g \in G$

$$\gamma(t + g) = \rho(g)\gamma(t) \tag{3.1}$$

for all $t \in S^1$.

Theorem 3.1. *Let $\mathbf{t}\colon S^1 \to S^{n-1}$ be a smooth curve so that the origin is in the interior of the convex hull of the image of $\mathbf{t}$ and assume that $\mathbf{t}$ has is G-symmetric with respect to some finite (and thus cyclic) subgroup of S^1. Then there is a G-symmetric regular closed curve $c\colon S^1 \to \mathbf{R}^n$ so that $\mathbf{t}(t) = c'(t)/\|c'(t)\|$.*

Proof. We first consider the case where G is a finite cyclic group. Let $\rho\colon G \to O(n)$ be so that (3.1) holds. From the proof of Theorem 1.2 we know that there is a function $v\colon \mathbf{R} \to (0, \infty)$ so that $\int_0^1 v(\tau)\mathbf{t}(\tau)\,d\tau = 0$. For any $g \in G$ we have

$$\begin{aligned}
\int_0^1 v(\tau - g)\mathbf{t}(\tau)\,d\tau &= \int_0^1 \mathbf{t}(\tau + g)\,d\tau \\
&= \int_0^1 v(\tau)\rho(g)\mathbf{t}(\tau)\,d\tau \\
&= \rho(g)\int_0^1 v(\tau)\mathbf{t}(\tau)\,d\tau \\
&= 0.
\end{aligned}$$

Therefore if $\tilde{v}$ is defined by

$$\tilde{v}(t) := \sum_{g\in G} v(t-g).$$

Then $\tilde{v}(t+g) = \tilde{v}(t)$ for all $g \in G$ (in particular $\tilde{v}(t+1) = \tilde{v}(t)$) and

$$\int_0^1 \tilde{v}(\tau)\mathbf{t}(\tau)\, d\tau = 0.$$

As in the proof of Theorem 1.2 the curve $c(t) := \int_0^t \tilde{v}(\tau)\mathbf{t}(\tau)\,\tau$ is a regular closed curve with tantrix $\mathbf{t}$. Also for $g \in G$

$$c'(t+g) = \tilde{v}(t+g)\mathbf{t}(t+b) = \tilde{v}(t)\rho(g)\mathbf{t}(t) = \rho(g)(\tilde{v}(t)\mathbf{t}(t)) = \rho(g)c'(t).$$

Therefore

$$\frac{d}{dt}(c(t+g) - \rho(g)c(t)) = c'(t+g) - \rho(g)c'(t) = \rho(g)c'(t) - \rho(g)c'(t) = 0.$$

This implies that for all $g \in G$ there is an $a(g) \in \mathbf{R}^n$ so that

$$c(t+g) = a(g) + \rho(g)c(t).$$

Lemma 3.2. *Let $A\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear map and let $a \in \mathbf{R}^n$ be so that for some positive integer $m \geq 2$ the equation*

$$(A^{m-1} + A^{m-1} + \cdots + A + I)a = 0 \tag{3.2}$$

holds. There there is a vector $b \in \mathbf{R}^n$ so that $(I-A)b = -a$.

Proof. We start with the polynomial identity

$$\begin{aligned}
(x-1)(x^{m-2} &+ 2x^{m-3} + 3x^{m-4} + \cdots + (m-2)x + (m-1)) \\
&= x^{m-1} + 2x^{m-2} + 3x^{m-3} + \cdots + (m-2)x^2 + (m-1)x \\
&\qquad - x^{m-2} - 2x^{m-3} - 3x^{m-4} + \cdots - (m-2)x - (m-1) \\
&= x^{m-1} + x^{m-2} + \cdots + x - (m-1) \\
&= (x^{m-1} + x^{m-2} + \cdots + x + 1) - m
\end{aligned}$$

Letting $p(x) = \frac{-1}{m}(x^{m-2} + 2x^{m-3} + 3x^{m-4} + \cdots + (m-2)x + (m-1))$ this leads to

$$-1 = \frac{-1}{m}(x^{m-1} + x^{m-2} + \cdots + x + 1) + (1-x)p(x).$$

Setting $x = A$ in this gives

$$-I = \frac{-1}{m}(A^{m-1} + A^{m-2} + \cdots + A + I) + (I-A)p(A).$$

Using the hypothesis (3.2) now gives that

$$-a = -Ia = (I-A)p(A)a.$$

Therefore $b = p(A)a$ is the desired vector. □

Let m be the order of the group G and let g be a generator of G. Let $a = a(g)$ so that $c(t+g) = a + \rho(g)c(t)$. Then

$$\begin{aligned} c(t+2g) &= a + \rho(g)c(t+g) = a + \rho(g)a + \rho(g)^2 c(t) \\ c(t+3g) &= a + \rho(g)c(t+2g) = a + \rho(g)a + \rho(g)^2 a + \rho(g)^3 c(t) \\ &\vdots \qquad\qquad \vdots \\ c(t+kg) &= a + \rho(g)a + \cdots + \rho(g)^{k-1}a + \rho(g)^k c(t). \end{aligned}$$

As the group G has order m we have $\rho(g)^m = I$ and so $\rho(g)^m c(t) = c(t)$. Therefore using $k = m$ in this calculation gives $(\rho(g)^{m-1} + \rho(g)^{m-1} + \cdots + \rho(g) + I)a = 0$. Therefore by Lemma 3.2 there is a vector b so that $(I - \rho(g))b = -a$. Now let $\tilde{c} = b + c(t)$. Then

$$\begin{aligned} \tilde{c}(t+g) &= b + c(t+g) \\ &= b + a + \rho(g)c(t) \\ &= (I - \rho(g))b + a + \rho(g)(b + c(t)) \\ &= 0 + \rho(g)\tilde{c}(t) \\ &= \rho(g)\tilde{c}(t) \end{aligned}$$

This and induction (or the calculation above with $a = 0$) implies that $\tilde{c}(t + kg) = \rho(g)^k \tilde{c}(t) = \rho(kg)\tilde{c}(t)$. As g is a generator of the group G every element of G is of the form kg for some k. This completes the proof. $\square$

References

1. W. Fenchel, *Uber krummungund windung geschlossener raumkurven*, Math. Ann. **101** (1929), 238–252.
2. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Dover Publications, New York, N. Y., 1945, Photographic reproduction of volumes 19 and 20 of the series Die Grundlehren der mathematischen Wissenschaften, published by J. Springer, Berlin, 1925.
3. R. Schneider, *Convex bodies: The Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, 1993.