ESTIMATES ON THE CONCENTRATION FUNCTION OF SETS IN $\mathbf{R}^d$: NOTES ON LECTURES OF OSKOLKOV

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1. Introduction

This set of notes is a result of a very pleasant interaction between the analysts, combinatorsts, and geometers in the mathematics department at the University of South Carolina. Let $v_1, \ldots, v_n$ be vectors in $\mathbf{R}^d$ so that any subset of $\{v_1, \ldots, v_n\}$ of size d is linearly independent (for convenience say that such a set is in ***general position***). Then Jerry Griggs has posed the problem of giving good upper and lower bounds on the number

$$\mathcal{C}(d,n) := \max_{v_1,\dots,v_n} \max_{y\in\mathbf{R}^d} \#\big\{(\varepsilon_1,\dots,\varepsilon_n) : \varepsilon_k = \pm 1 \text{ and } \sum_{k=1}^n \varepsilon_k x_k = y\big\}$$

where $v_1,\dots,v_n$ varies sets in general position. By using a clever combination of linear algebra, probability, and elementary number theory Griggs [1] was given a lower bound

$$\mathcal{C}(d,n) \ge C(d)\left(\frac{1}{n\sqrt{n}}\right)^d 2^n.$$

In the paper [2] Halász gives upper bounds for a more general problem with imply that

$$\mathcal{C}(d,n) \le C'(d)\frac{1}{n}\left(\frac{1}{\sqrt{n}}\right)^d 2^n.$$

Halász's paper uses quite a bit of machinary from probability theory and in lectures in our combintoric's seminar Kostya Oskolkov showed that the probabality could be avoided and that Halász's result can be reduced to estimating certian integrals whoses integrands are products of absolute values of cosines. My contribution to this project is noting that in some cases these integrals can be estimated by use of rearrangment inequalites. This very much simplifies the proof of Thoerm 1 in [2]. However this only leads to the upper bound

$$\mathcal{C}(d,n) \le C''(d)\left(\frac{1}{\sqrt{n}}\right)^d 2^n.$$

Unfortunately there seem to be structural reasons why the simplier methods used in the notes here can not give the stronger result.

The notes here are basically an expanded version of Oskolkov's lectures with some proofs and references added on the use of rearrangement inequalities (the basic reference here is the wonderful book [3] of Leib and Loss). The main result here (Theorem 4.3) is a slight refinement of Theorem 1 of [2] in that all constants are given explicitly[1]. I hope that these notes are readable to non-analysts and in particular to combinatorsts.

2. Statement of the problem and application of the Fourier transform.

2.1. **The Problem.** Let $v_1,\dots,v_n \in \mathbf{R}^d$ be n vectors in the d dimensional Euclidean space $\mathbf{R}^d$ and $D \subset \mathbf{R}^d$ a bounded domain and define

$$N_D = \#\big\{(\varepsilon_1,\dots,\varepsilon_n) : \varepsilon_k = \pm 1 \text{ and } \sum_{k=1}^n \varepsilon_k v_k \in D\big\}.$$

[1] Note however that "explicit" does not mean "correct". This is a technical subject and the amount of proof reading required to insure the correctness of all the constants was more than I was up for.

Then N_D is the number, counted with multiplicity, of the sums $\pm v_1 \pm \cdots \pm v_n$ that are in D. Thus different choice of $\varepsilon_1, \ldots, \varepsilon_n$ that give rise to the same sum in D each count in computing D. In the extreme case where $v_k = 0$ for all k all the sums $\sum_{k=1}^n \varepsilon_k v_k$ are the same and equal to 0, but in this case if $0 \in D$ we have $N_D = 2^n$. The translate of D by $y \in \mathbf{R}^d$ is $D + y = \{x + y : x \in D\}$. Define $\mathcal{N}_D$ by

$$\mathcal{N}_D = \max_{y \in \mathbf{R}^d} N_{D+y}$$

The number $\mathcal{N}_D$ is thus just the maximum number of sums $\sum_{k=1}^n \varepsilon_k v_k$ (counted with multiplicity) that can be captured by a translate of D and is called the ***concentration***. We wish to give a method for estimating $\mathcal{N}_D$ form above in terms of the geometry of D and $v_1, \ldots, v_n$. We will start by estimating N_D, but the resulting inequality will end up being "translation invariant" and thus also gives a bound on $\mathcal{N}_D$.

2.2. **Reduction to an analytic problem.** We first give an integral formula for N_D using a trick from analytic number theory. For any subset $A \subset \mathbf{R}$ denote the Lebesgue measure of A by $|A|$. Let $r_1, \ldots, r_n \colon [0,1] \to \mathbf{R}$ be independent random variables all with the distribution that takes on the values ± 1 each with probability $1/2$. (Those wishing to avoid all use of probability see Remark 2.3 below.) A concrete choice of $r_1, \ldots, r_n$ is the ***Rademacher functions*** given by $r_k(t) = \operatorname{sign} \sin(2^k \pi t)$ in which case it is easy to check that

$$|\{t \in [0,1] : r_k(t) = +1\}| = |\{t \in [0,1] : r_k(t) = -1\}| = \frac{1}{2}.$$

It takes a little more work to show these are independent as random variables, but this is still elementary. Let

$$S(t) := \sum_{k=1}^m r_k(t) x_k.$$

Then, because each $r_k(t)$ takes on the values ± 1 and the r_k are independent random variables, as t ranges over $[0,1]$ the function $S(t)$ ranges over all the sums $\sum \varepsilon_k x_k$ and for a give choice of $\varepsilon_1, \ldots, \varepsilon_n$ the measure of the set of t that realizes this sum is $1/2^n$. Denote by χ_D is the characteristic function of D (i.e. $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ if $x \notin D$) then from the properties of $S(t)$ just given

$$N_D = 2^n \int_0^1 \chi_D(S(t))\, dt.$$

If $\varphi \colon \mathbf{R}^d \to \mathbf{R}$ is a function that satisfies

$$\varphi \geq \chi_D$$

then there is the obvious inequality

$$N_D \leq 2^n \int_0^1 \varphi(S(t))\, dt. \tag{2.1}$$

The reason we wish to replace χ_D a function $\varphi \geq \chi_D$ is that we will be using Fourier transform methods and the Fourier transform of χ_D is rather unpleasant (for example it is not in $L^1(\mathbf{R}^d)$) and we will be able to make choices of φ whose Fourier transforms are easier to work with.

Our convection on the constants in the Fourier transform are as follows. If $f \in L^1(\mathbf{R}^d)$ then its Fourier transform is

$$\widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi\rangle}\, dx$$

where $\langle\, ,\rangle$ is the standard inner product on $\mathbf{R}^d$ and dx is the standard volume measure. If f and $\widehat{f}$ are both in L^1 then the Fourier inversion formula holds and is given by

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \widehat{f}(\xi) e^{i\langle x, \xi\rangle}\, d\xi.$$

We now let $\varphi \geq \chi_D$ be a function so that both φ and $\widehat{\varphi}$ are in L^1 and use the Fourier inversion formula in the inequality (2.1) and invert the order of integration to get

$$N_D \leq 2^n (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \widehat{\varphi}(\xi) \int_0^1 e^{i\langle S(t), \xi\rangle}\, dt\, d\xi. \tag{2.2}$$

To simplify this note that as each $r_k(t)$ takes on the values $+1$ and -1 each with measure $1/2$ for any real number c

$$\int_0^1 e^{ir_k(t)c}\, dt = \frac{1}{2}e^{ic} + \frac{1}{2}e^{-ic} = \cos(c).$$

Using this, the definition of $S(t)$, and the independence of $r_1, \ldots, r_n$

$$\begin{aligned}
\int_0^1 e^{i\langle S(t),\xi\rangle}\, dt &= \int_0^1 e^{i(r_1(t)\langle x_1,\xi\rangle + \cdots + r_n(t)\langle x_n,\xi\rangle)} dt \\
&= \int_0^1 \prod_{k=1}^n e^{ir_k(t)\langle x_k,\xi\rangle} dt \\
&= \prod_{k=1}^n \int_0^1 e^{ir_k(t)\langle x_k,\xi\rangle} dt \\
&= \prod_{k=1}^n \cos\langle x_k, \xi\rangle. \qquad (2.3)
\end{aligned}$$

Putting this back in (2.2) gives

$$\begin{aligned}
N_D &\leq (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \widehat{\varphi}(\xi) \prod_{k=1}^n \cos\langle x_k, \xi\rangle\, d\xi \\
&\leq (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} |\widehat{\varphi}(\xi)| \prod_{k=1}^n |\cos\langle x_k, \xi\rangle|\, d\xi \qquad (2.4)
\end{aligned}$$

Thus we have done most of the work toward proving:

Proposition 2.1 (Oskolkov's Lemma). *Let D be a bounded domain in $\mathbf{R}^d$ and φ a function so that $\varphi \geq \chi_D$ and both φ and $\widehat{\varphi}$ are in $L^1(\mathbf{R}^d)$. Then*

$$\mathcal{N}_D \leq (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} |\widehat{\varphi}(\xi)| \prod_{k=1}^{n} |\cos\langle x_k, \xi\rangle| \, d\xi. \tag{2.5}$$

Proof. For any $y \in \mathbf{R}^d$ and function $f\colon \mathbf{R}^d \to \mathbf{R}$ let $\tau_y f(x) := f(x-y)$ be the translate of f by y. Then $\tau_y \chi_D = \chi_{D+y}$. Let $\varphi \geq \chi_D$ as in the statement of the proposition. Then $\tau_y \varphi \geq \chi_{D+y}$. The Fourier transform of a translation is given by $\widehat{\tau_y \varphi}(\xi) = e^{-i\langle \xi, y\rangle}\widehat{\varphi}(\xi)$ and therefore $|\widehat{\tau_y\varphi}(\xi)| = |\widehat{\varphi}(\xi)|$. Using this in (2.4) gives

$$N_{D+y} \leq (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} |\widehat{\varphi}(\xi)| \prod_{k=1}^{n} |\cos\langle x_k, \xi\rangle| \, d\xi.$$

The right side of this is independent of y and (2.5) follows. □

Remark 2.2. We are now in the situation of many proofs in analysis. We have the inequality (2.5) which depends on an arbitrary "test function" φ. The game now becomes to find a good, or at least a manageable choice, of φ. We will start by looking at choices for φ in the one dimensional case and then using these to get construct φ in higher dimensions in the case D is a cube. □

Remark 2.3. The basic property of the functions $r_k(t)$ used was the computation 2.3. Using the Rademacher functions this calculation can be directly form the definition of the $r_k(t)$'s and all mention of probability thus avoided. □

2.3. **A choice of the test function φ in one dimension and for D an interval.** First we review a little about the Fourier transform in for functions on $\mathbf{R}^d$. If $f, g \in L^1(\mathbf{R}^d)$ then the convolution $f * g$ of f and g is given by

$$f * g(x) = \int_{\mathbf{R}^d} f(x-y)g(y)\, dy = \int_{\mathbf{R}^d} f(y)g(x-y)\, dy.$$

This is also in $L^1(\mathbf{R}^d)$. The Fourier transform and convolution are related by

$$(2\pi)^{-d/2}\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

(The factor on the left is due to the choice of where we are putting the 2π's in the definition of the Fourier transform.) Back in the one dimensional case let $f(\xi) = \chi_{[-a,a]}(\xi)$ be the characteristic function of an interval of length $2a$ centered at the origin. Then

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ix\xi} d\xi = \sqrt{\frac{2}{\pi}}\, \frac{\sin(ax)}{x}.$$

Then

$$\frac{1}{\sqrt{2\pi}}\widehat{f*f}(x) = \widehat{f}(x)^2 = \frac{2}{\pi}\frac{\sin^2(ax)}{x^2}.$$

The smallest positive zero of $\sin(ax)/x$ is at $x = \pi/a$ and the function $\sin(ax)/x$ is decreasing on $[0, \pi/a]$. Let $r = \pi/a$ so that $a = \pi/r$ and $r/2 = \pi/(2a)$. For these values of r and a the smallest values of $\sin^2(ax)/x^2$ on the interval $[-r/2, r/2]$ occur when $x = \pm r/2$ and the smallest value is

$$\frac{\sin^2(ra/2)}{(r/2)^2} = \frac{4}{r^2}.$$

Therefore

$$\frac{\sqrt{2\pi}r^2}{16}\widehat{f*f}(x) = \frac{r^2\sin^2(\pi x/r)}{4x^2} \geq 1 \quad \text{for} \quad x \in [-r/2, r/2].$$

We also can compute $f * f$ explicitly and get

$$f * f(\xi) = \max(0, 2a - |\xi|) = \max(0, 2\pi/r - |\xi|^2).$$

Lemma 2.4. *If* $\varphi_1 \colon \mathbf{R} \to [0,\infty)$ *is given by*

$$\varphi_1(x) = \frac{r^2\sin^2(\pi x/r)}{4x^2}$$

then $\varphi_1 \geq \chi_{[-r/2,r/2]}$, $\varphi_1 \in L^1(\mathbf{R})$, *and*

$$\widehat{\varphi_1}(\xi) = \frac{\sqrt{2\pi}r^2}{16}\max(0, 2\pi/r - |\xi|)$$

which is supported in $[-2\pi/r, 2\pi/r]$.

Proof. On the set of functions symmetric about the origin (that is $f(-x) = f(x)$ the Fourier transform and its inverse are given by the same formula. Therefore applying the inverse transform to the calculations above gives the result. □

2.4. **The test function for D a cube in $\mathbf{R}^d$.** Let $Q(r)$ be the cube $[-r/2, r/2]^d$ be the centered cube in with side of length r in $\mathbf{R}^d$, and let $x_1, \dots, x_d$ be the coordinate functions on $\mathbf{R}^d$.

Lemma 2.5. *Define a function* $\varphi \colon \mathbf{R}^d \to [0,\infty)$ *by*

$$\varphi(x) = \left(\frac{r^2}{4}\right)^d \prod_{j=1}^d \frac{\sin^2(\pi x_j/r)}{x_j^2}$$

then $\varphi \in L^1(\mathbf{R}^d)$, $\varphi \geq \chi_{Q(r)}$,

$$\widehat{\varphi}(\xi) = \left(\frac{\sqrt{2\pi}r^2}{16}\right)^d \prod_{j=1}^d \max(0, 2\pi/r - |\xi_j|)$$

which is supported in $Q(2\pi/r)$ and satisfies the bound

$$|\widehat{\varphi}(\xi)| \leq \left(\frac{(2\pi)^{3/2} ra}{16}\right)^d .$$

Proof. With the notation of Lamma 2.4 we have $\varphi(x) = \varphi_1(x_1)\varphi_1(x_2)\cdots\varphi_1(x_d)$ and the results follow from that lemma. □

Proposition 2.6. *Let $Q(r)$ be the cube with sides parallel to the coordiante axis in $\mathbf{R}^d$ and edge lenght of r. Then the concentration of $v_1, \ldots, v_k$ satisfies*

$$\mathcal{N}_{Q(r)} \leq 2^n \left(\frac{\pi r}{8}\right)^d \int_{Q(2\pi/r)} \prod_{k=1}^n |\cos\langle v_k, \xi\rangle| \, d\xi.$$

Proof. We use the test function φ of Lemma 2.5 (and the bounds given there) in Proposition 2.1 to get:

$$\begin{aligned}
\mathcal{N}_{Q(r)} &\leq (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} |\widehat{\varphi}(\xi)| \prod_{k=1}^n |\cos\langle v_k, \xi\rangle| \, d\xi \\
&\leq 2^n (2\pi)^{-\frac{d}{2}} \left(\frac{(2\pi)^{3/2} ra}{16}\right)^d \int_{Q(2\pi/2)} \prod_{k=1}^n |\cos\langle v_k, \xi\rangle| \, d\xi \\
&= 2^n \left(\frac{\pi r}{8}\right)^d \int_{Q(2\pi/r)} \prod_{k=1}^n |\cos\langle v_k, \xi\rangle| \, d\xi
\end{aligned}$$

This completes the proof. □

Remark 2.7. To get an idea if this estimate is any good we look at the extreme case where $v_k = 0$ for all k. Then $\mathcal{N}_D = 2^n$ for any D. The estimate of the last proposition gives (using that $\cos\langle v_k, \xi\rangle = \cos(0) = 1$)

$$\mathcal{N}_{Q(r)} \leq 2^n \left(\frac{\pi r}{8}\right)^d \mathrm{Vol}(Q(2\pi/r)) = 2^n \left(\frac{\pi^2}{4}\right)^d \leq 2^n (2.468)^d$$

which is not all that good, but not outrageously bad. □

3. Rearrangement inequalities and their applications.

3.1. **The basic rearrangement inequalities.** Let $A \subset \mathbf{R}^d$ be a Lebesgue measurable set and denote the measure of A by $|A|$. Define the ***symmetirzation*** A^* of A to be the closed ball centered at the origin such with the same measure as A. Thus in one dimension

$$A^* := [-|A|/2, |A|/2]$$

and in two dimensions

$$A^* := B(0, (|A|/\pi)^{1/2}).$$

More generally if define $\omega(d)$ to be the volume of the unit ball in $\mathbf{R}^d$ then for $A \subset \mathbf{R}^d$

$$A^* := B(0, (|A|/\omega(d))^{1/d}).$$

We now wish to extend this idea of syemmtrization from sets to non-negative functions. For a charactertic function χ_A of a set how to do this is clear:

$$\chi_A^* := \chi_{A^*}$$

We now use that any non-negative function can be expressed as an integral of the characteristic functions of the sets $\{f \geq t\}$ (which is a standard abbreviation for $\{x : f(x) \geq t\}$) as follows

$$f(x) = \int_0^{f(x)} 1\, dt = \int_0^\infty \chi_{\{f \geq t\}}(x)\, dt. \tag{3.1}$$

This representation of f is sometimes called the ***layer cake representation*** of f. Note that this, along with Fubini's theorem, implies

$$\begin{aligned} \int_{\mathbf{R}^d} f(x)\, dx &= \int_{\mathbf{R}^d} \int_0^\infty \chi_{\{f \geq t\}}(x)\, dt\, dx \\ &= \int_0^\infty \chi_{\{f \geq t\}}(x)\, dx\, dt \\ &= \int_0^\infty |\{x : f(x) \geq t\}|\, dt. \end{aligned}$$

We now define the ***symmetric decreasing rearrangement*** of a non-negative measurable function f by

$$f^*(x) := \int_0^\infty \chi^*_{\{f \geq t\}}(x)\, dt. \tag{3.2}$$

The basic properties of f^* are that it is monotone decreasing as a function of $|x|$:

$$|x| \leq |y| \quad \text{implies} \quad f^*(x) \geq f^*(y). \tag{3.3}$$

Letting $|x| = |y|$ (so that $|x| \leq |y|$ and $|y| \leq |x|$) this implies f^* is symmetric

$$|x| = |y| \quad \text{implies} \quad f^*(x) = f^*(y).$$

A little less obvious but very important is that f and f^* are equi-measurable in the sense that

$$|\{f^* \geq t\}| = |\{f \geq t\}| \tag{3.4}$$

for all t.

The two properties (3.3) and (3.4) come close to characterizing f^* and in fact do characterize f^* up to a set of measure zero.

What makes the symmetric decreasing rearrangement interesting to us is

Theorem 3.1 (Basic rearrangement inequality)**.** *Let $f_1, \ldots, f_m \colon \mathbf{R}^d \to [0, \infty)$ be non-negative measurable functions. Then*

$$\int_{\mathbf{R}^d} f_1(x) f_2(x) \cdots f_m(x)\, dx \leq \int_{\mathbf{R}^d} f_1^*(x) f_2^*(x) \cdots f_m^*(x)\, dx.$$

Proof. We first consider the case where each $f_k = \chi_{A_k}$ is the characteristic function of a measurable set. By reordering we can assume that A_1 has the smallest measure of the sets $A_1, \dots, A_m$. Then the product $f_1 f_2 \cdots f_m$ is just $\chi_{A_1\cap\cdots\cap A_m}$, that is the characteristic function of the intesection $A_1 \cap \cdots \cap A_m$. But $A_1 \cap \cdots \cap A_m \subseteq A_1$ and so

$$\int_{\mathbf{R}^d} f_1(x) f_2(x) \cdots f_m(x)\, dx = |A_1 \cap \cdots \cap A_m| \le |A_1| \tag{3.5}$$

But in this case $f_k^* = \chi_{A_k^*}$ and $|A_k^*| = |A_k|$. Thus A_1^* will have the smallest measure of the sets $A_1^*, \dots, A_m^*$. But as all the A_k^* are closed balls centered at the origin this implies $A_1^* \cap \cdots \cap A_m^* = A_1^*$. Thus

$$\int_{\mathbf{R}^d} f_1^*(x) f_2^*(x) \cdots f_m^*(x)\, dx = |A_1^* \cap \cdots \cap A_m^*| = |A_1^*| = |A_1|.$$

Putting this together with (3.5) implies the result in the case all the f_k are characteristic functions. In the general case we use the layer cake representation (3.1) to reduce to the case of characteristic functions. In the following we interchange the order of integration without comment.

$$\begin{aligned}
\int_{\mathbf{R}^d} & f_1(x) f_2(x) \cdots f_m(x)\, dx \\
&= \int_0^\infty \cdots \int_0^\infty \int_{\mathbf{R}^d} \chi_{\{f_1 \ge t_1\}}(x) \cdots \chi_{\{f_m \ge t_m\}}(x)\, dx\, dt_1 \cdots dt_m \\
&\le \int_0^\infty \cdots \int_0^\infty \int_{\mathbf{R}^d} \chi_{\{f_1^* \ge t_1\}}(x) \cdots \chi_{\{f_m^* \ge t_m\}}(x)\, dx\, dt_1 \cdots dt_m \\
&= \int_{\mathbf{R}^d} f_1^*(x) f_2^*(x) \cdots f_m^*(x)\, dx.
\end{aligned}$$

This completes the proof. □

3.2. **A trick for reducing the dimension of the problem.** The next result is basically just a technical lemma that shows us in some cases to reduce a problem in estimating a function of one function of several variables to a several (hopefully easier) estimates of functions of one variable. First some notation. For $x \in \mathbf{R}^d$ let $x_1, \dots, x_d$ be the coordinates of x (so that $x = (x_1, \dots, x_d)$). Then for $1 \le j \le d$ and $f(x) = (x_1, \dots, x_d)$ let

$$\mathcal{S}_j^* f(x_1, \dots, x_d) = \begin{cases} \text{symmetric decreasing rearrangment of } f \text{ with} \\ \text{respect to } x_j \text{ holding other variables fixed.} \end{cases}$$

This one variable version of symmetrization is called ***Steiner symmetrization*** after Steiner who introduced it early in the last century and who seems to have been to use symmetrization methods in proving geometric inequalities.

Proposition 3.2. *Let $F\colon \mathbf{R}^d \to [0, \infty)$ be a nonnegative measurable function that can be factored as*

$$F(x) = F(x_1, \dots, x_d) = f_1(x_1) f_2(x_1, x_2) f_3(x_1, x_2, x_3) \cdots f_d(x_1, \dots, x_d)$$

where f_j only depends on $x_1, \dots, x_j$. Assume that for each j there is a symmetric deceasing function $g_j(t)$ of one variable so that

$$\mathcal{S}_j^* f_j(x_1, \dots, x_j) \le g_j(x_j)$$

and also assume that there is a symmetric decreasing function $G\colon \mathbf{R}^d \to [0, \infty)$ so that

$$g_1(x_1)g_2(x_2)\cdots g_d(x_d) \le G(x_1, \dots, x_d) = G(x).$$

Then

$$F^*(x) \le G(x)$$

for all x.

Proof. It is not hard to verify from the definitions that

$$\begin{aligned}
|\{x : F(x) \ge t\}| &= |\{(x_1, \dots, x_d) : f_1(x_1)\cdots f_d(x_1, \dots, x_d) \ge t\}| \\
&\le |\{(x_1, \dots, x_d) : f_1(x_1)\cdots f_{d-1}(x_1, \dots, x_{d-1})g_d(x_d) \ge t\}| \\
&\le |\{(x_1, \dots, x_d) : f_1(x_1)\cdots f_{d-2}(x_1, \dots, x_{d-2})g_{d-1}(x_{d-1})g_d(x_d) \ge t\}| \\
&\ \ \vdots \\
&\le |\{(x_1, \dots, x_d) : g_1(x_1)\cdots g_{d-1}(x_{d-1})g_d(x_d) \ge t\}| \\
&\le |\{(x_1, \dots, x_d) : G(x_1, \dots, x_d) \ge t\}|.
\end{aligned}$$

And so $F^*(t) \le G^*(t) = G(t)$ (where $G^* = G$ as G is symmetric and a monotone decreasing function of $|x|$). This completes the proof. □

3.3. **Application of the rearrangement inequalities to integrals of products of cosines.**

Lemma 3.3. *Let $f : [-R, R] \to [0, 1]$ be given by $f(x) = |\cos(ax + b)|$ where a is an integral multiple of $\pi/(2R)$ (this implies that $2R$ is a period of $f(x)$)). Then*

$$f^*(x) = \cos(\pi x/(2R)).$$

Proof. Doing this in terms of the definition (3.2) is hard work, but by using the characterization (3.3) and (3.4) and drawing a picture this becomes more or less clear. □

Lemma 3.4. *Let $f : [-R, R] \to [0, 1]$ be given by $f(x) = |\cos(ax+b)|$ where $a \ge \pi/(2R)$. Then*

$$f^*(x) \le \cos(\pi x/(4R)).$$

Proof. As $a \ge \pi/(2R)$ the smallest $P_{\min}$ period of $f(x)$ is $\le 2R$. Let R_1 be the smallest number $\ge R$ so that $2R_1$ is a period of $f(x)$. Then $R_1 \le R$. Let $f_1(x)$ be the natural extension of $f(x)$ to $[-R_1, R_1]$ (that is $f_1(x) = |\sin(ax+b)|$ for $|x| \le R_1$). From the characterization of symmetric rearrangements given by (3.3) and (3.4) we clearly have $f^*(x) \le f_1^*(x)$ for $|x| \le R$. But then the last lemma and $R_1 \le R$ implies

$$f^*(x) \le f_1^*(x) = \cos(\pi x/(2R_1)) \le \cos(\pi x/(4R)).$$

This completes the proof. □

Lemma 3.5. *Let* $f : [-R, R] \to [0, 1]$ *be given by* $f(x) = |\cos(ax+b)|$ *where* $a \leq \pi/(2R)$. *Then*

$$f^*(x) \leq \cos(ax).$$

Proof. An exercise. □

Lemma 3.6. *For* $|t| \leq \pi/2$ *the inequality*

$$\cos(t) \leq e^{-t^2/2}$$

holds.

Proof. Anther exercise. □

Lemma 3.7. *Let* $f_1, \ldots, f_d \colon [-R, R]^d \colon \to \mathbf{R}$ *be* d *functions given by*

$$f_j(x_1, \ldots, x_j) = |\cos(a_j x_j + b_j(x_1, \ldots, x_{d-1}))|.$$

(thus f_j *only depends on the variables* $x_1, \ldots, x_j$ *and* b_j *only depends on* $x_1, \ldots, x_{j-1}$*). Let* $F(x) = F(x_1, \ldots, x_d)$ *be*

$$F(x) = f_1(x) \cdots f_d(x)$$

the product of the f_j. *Assume that* $|a_j| \geq \pi/(2R)$. *Then*

$$F^*(x_1, \ldots, x_d) \leq e^{-\frac{1}{2}\frac{\pi^2}{16R^2}(x_1^2+\cdots+x_d^2)}$$

Remark 3.8. In terms of other notation used here $[-R, R]^d = Q(2R)$. □

Proof. Let $g_j(t) = |\cos(\pi t/(4R))|$ Then by Lemma 3.4 the inequality

$$\mathcal{S}_j^* f_j(x_1, \ldots, x_j) \leq g_j(x_j) = |\cos(\pi x_j/(4R))|.$$

As $|x_j| \leq R$ we have $|\pi x_j/(4R)| \leq \pi/4 < \pi/2$ and this by Lemma 3.6 we can estimate the product

$$g_1(x_1)g_2(x_2)\cdots g_d(x_d) \leq e^{\frac{-\pi^2 x_1^2}{32R^2}} e^{\frac{-\pi^2 x_2^2}{32R^2}} \cdots e^{\frac{-\pi^2 x_d^2}{32R^2}} = e^{-\frac{1}{2}\frac{\pi^2}{16R^2}(x_1^2+\cdots+x_d^2)}.$$

The function $G(x_1, \ldots, x_d) = e^{-\frac{1}{2}\frac{\pi^2}{16R^2}(x_1^2+\cdots+x_d^2)}$ is symmetric and decreasing so that Proposition 3.2 applies. This completes the proof. □

Proposition 3.9. *Let* $v_1, \ldots, v_n \in \mathbf{R}^d$ *be* n *vectors in* $\mathbf{R}^d$ *and let* $m \leq n$. *Assume that for any unit vector* $u \in \mathbf{R}^d$ *that for at least* m *of the indices* k *the inequality*

$$|\langle v_k, u \rangle| \geq \pi/(2\sqrt{d}R)$$

holds. Then

$$\int_{Q(2R)} \prod_{k=1}^{n} |\cos\langle \xi, v_k \rangle| \, d\xi \leq \left(\frac{8dR}{\sqrt{2\pi}}\right)^d \left(\frac{1}{\sqrt{m}}\right)^d$$

Proof. Let call an ordered set $(v_{k_1}, \dots, v_{k_d})$ from the list $v_1, \dots, v_n$ ***nice*** iff the vectors $v_{k_1}, \dots, v_{k_d}$ are linearly independent, $|v_1| \ge \pi/(2\sqrt{d}R)$ and for $2 \le j \le d$ the length of the orthogonal projection of v_{k_j} onto $\operatorname{span}(v_{k_1}, \dots, v_{k_{j-1}})^\perp$ has length $\ge \pi/(2\sqrt{d}R)$. If the number m in the statement of the proposition is $\ge d$ then we can construct a nice ordered set $(v_{k_1}, \dots, v_{k_d})$ as follows. First choose v_{k_1} so that $|v_{k_1}| \ge \pi/(2\sqrt{d}R)$ and for future use let $e_1 := v_{k_1}/|v_{k_1}|$ be the unit vector in the direction of v_{k_1}. Now assume that $v_{k_1}, \dots, v_{k_{j-1}}$ have been defined. Then let $u \in \operatorname{span}(v_{k_1}, \dots, v_{k_{j-1}})^\perp$ be a unit vector. Then there is a vector v_{k_j} so that $|\langle v_{k_j}, u\rangle| \ge \pi/(2\sqrt{d}R)$. This imples that if $v_{k_j}^\perp$ is the orthogonal projection of v_{k_j} onto $\operatorname{span}(v_{k_1}, \dots, v_{k_{j-1}})^\perp$ then $|v_{k_j}^\perp| \ge \pi/(2\sqrt{d}R)$. Now let $e_k := v_{k_j}^\perp/|v_{k_j}^\perp|$ be the unit vector in the direction of $v_{k_j}^\perp$. Then $(v_{k_1}, \dots, v_{k_d})$ is nice and also $e_1, \dots, e_d$ is an orthonormal basis of $\mathbf{R}^d$ so that

$$\operatorname{span}(e_1, \dots, e_j) = \operatorname{span}(v_{k_1}, \dots, v_{k_j}) \quad \text{and} \quad |\langle e_j, v_{k_j}\rangle| \ge \pi/(2\sqrt{d}R) \tag{3.6}$$

for $j = 1, \dots, d$.

Let p be the largest integer so that $pd \le m$ (thus $p \le m/d$). Then be doing the construction above p times we can find p nice ordered sets with no elements in common and be reordering we can assume that the nice ordered sets are

$$(v_1, \dots, v_d), (v_{d+1}, \dots, v_{2d}), \dots, (v_{\ell d+1}, \dots, v_{(\ell+1)d}), \dots, (v_{(p-1)d+1}, \dots, v_{pd}).$$

Let F_ℓ be the function

$$F_\ell(\xi) = |\cos\langle \xi, v_{\ell d+1}\rangle||\cos\langle \xi, v_{\ell d+2}\rangle| \cdots |\cos\langle \xi, v_{(\ell+1)d}\rangle|.$$

Then

$$\prod_{k=1}^{n} |\cos\langle \xi, v_k\rangle| = \prod_{\ell=1}^{p} F_\ell(x) \cdot \prod_{k=pd+1}^{n} |\cos\langle \xi, v_k\rangle| \le \prod_{\ell=1}^{p} F_\ell(x) \tag{3.7}$$

We now estimate the symmetric rearrangement of F_1. Let $e_1, \dots, e_d$ be the orthonormal basis of $\mathbf{R}^d$ associated with the nice ordered set $v_1, \dots, v_d$ as in (3.6). Then do an orthogonal change of variables $\eta = P\xi$ (thus P is an orthogonal matrix) so that in the variables $\eta = (\eta_1, \dots, \eta_d)$ the η_j axis is in the direction of e_j. Let $Q'(2R)$ be the image of the cube $Q(2R)$ under this change of variables. Then $Q'(2R) \subset Q(2\sqrt{d}R)$ (where the cube $Q(2\sqrt{d}R)$ is defined with respect to η coordinate system). Let $F'_\ell(\eta)$ be F_ℓ in the new coordinate system (and we extend F'_ℓ to $Q(2\sqrt{d}R)$ so that in $Q(2\sqrt{d}R)$ is still given by the same formulas as in $Q'(2R)$). Now in the coordinate system $\eta_1, \dots, \eta_d$ we have

$$F'_1(\eta) = \prod_{j=1}^{d} |\cos(a_j\eta_j + b_j(\eta_1, \dots, \eta_{j-1}))|$$

where $|a_j| = |\langle v_j, e_j\rangle| \geq \pi/(2\sqrt{d}R)$. Therefore by Lemma 3.7 (with R replaced by $\sqrt{d}R$) we have

$$(F_1')^*(\eta_1,\dots,\eta_d) \leq e^{-\frac{1}{2}\frac{\pi^2}{16\sqrt{d}R^2}|\eta|^2}$$

Translating this back to the original coordinate system gives

$$F_1^* \leq e^{-\frac{1}{2}\frac{\pi^2}{16\sqrt{d}R^2}|\eta|^2}.$$

Clearly the same estimate will hold for F_ℓ with $1 \leq \ell \leq p$. Thus by the basic rearrangement inequality we have

$$\begin{aligned}
\int_{Q(2R)} \prod_{\ell=1}^{p} F_\ell(\xi)\,d\xi &\leq \int_{\mathbf{R}^d} \prod_{\ell=1}^{p} F_\ell^*(\xi)\,d\xi \\
&\leq \int_{\mathbf{R}^d} e^{-\frac{1}{2}\frac{m\pi^2}{16\sqrt{d}R^2}|\eta|^2}\,d\xi \\
&= (\sqrt{2\pi})^d \left(\frac{1}{\sqrt{(p\pi^2)/(16dR^2)}}\right)^d \\
&\leq \left(\frac{8dR}{\sqrt{2\pi}}\right)^d \left(\frac{1}{\sqrt{m}}\right)^d.
\end{aligned}$$

Combining this with (3.7) completes the proof. □

Proposition 3.10. *Let $a_1,\dots,a_n$ be non-negative real numbers and $R > 0$. Let m be the number of the a_k's that satisfy $a_k > \pi/(2R)$. By reordering it can be assume that $a_k \leq \pi/(2R)$ for $k \leq (n-m)$ and $a_k > \pi/(2R)$ for $k \geq (n-m+1)$. Then*

$$\int_{-R}^{R} \prod_{k=1}^{n} |\cos(a_k x + b_k)|\,dx \leq \frac{\sqrt{2\pi}}{\sqrt{(m\pi^2)/(16R^2) + \sum_{k=1}^{n-m} a_k^2}}.$$

Thus

$$\int_{-R}^{R} \prod_{k=1}^{n} |\cos(a_k x + b_k)|\,dx \leq \frac{\sqrt{2\pi}}{\sqrt{(m\pi^2)/(16R^2)}} = \frac{8R}{\sqrt{2\pi m}}$$

and

$$\int_{-R}^{R} \prod_{k=1}^{n} |\cos(a_k x + b_k)|\,dx \leq \frac{\sqrt{2\pi}}{\sqrt{\sum_{k=1}^{n-m} a_k^2}}.$$

Proof. From the rearrangement Theorem 3.1 and Lemmas 3.4 and 3.5 which give explicit bounds on the symmetric rearrangements of the functions $f_k(x) = |\cos(a_k x + b_k)|$ we have

$$\int_{-R}^{R} \prod_{k=1}^{n} |\cos(a_k x + b_k)|\,dx \leq \int_{-R}^{R} \cos^m(\pi x/(4R)) \prod_{k=1}^{n-m} \cos(a_k x)\,dx.$$

Now use the inequality

$$\cos(x) \le e^{-x^2/2} \quad \text{for} \quad |x| \le \pi/2$$

of Lemma 3.6 to estimate

$$\int_{-R}^{R} \cos^m(\pi x/(4R)) \prod_{k=1}^{n-m} \cos(a_k x)\, dx \le \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{m\pi^2}{16R^2}+\sum_{k=1}^{n-m} a_k^2\right)x^2}\, dx$$
$$= \frac{\sqrt{2\pi}}{\sqrt{(m\pi^2)/(16R^2)+\sum_{k=1}^{n-m} a_k^2}}.$$

Putting these inequalities together completes the proof. □

4. Estimates on the concentration.

4.1. A one dimensional result.

Theorem 4.1. *Let $a_1, \dots, a_n$ be real numbers and let m be the number of the a_k's that satisfy $|a_k| > 1$ and ordered so that $|a_k| \ge 1$ for $k > n-m$. Then*

$$\mathcal{N}_{[-1,1]} \le \frac{\pi\sqrt{2\pi}}{2\sqrt{m+4\sum_{k=1}^{n-m} a_k^2}}\, 2^n \le \frac{3.94}{\sqrt{m+4\sum_{k=1}^{n-m} a_k^2}}\, 2^n.$$

Thus

$$\mathcal{N}_{[-1,1]} \le \frac{\pi\sqrt{2\pi}}{2\sqrt{m}}\, 2^n \le \frac{3.94}{\sqrt{m}}\, 2^n$$

and

$$\mathcal{N}_{[-1,1]} \le \frac{\pi\sqrt{2\pi}}{4\sqrt{\sum_{k=1}^{n-m} a_k^2}}\, 2^n \le \frac{1.97}{\sqrt{\sum_{k=1}^{n-m} a_k^2}}\, 2^n.$$

Proof. Replacing a_k by $-a_k$ does not change the value of $\mathcal{N}_{[-1,1]}$ and so there is no loss in generality by assuming that $a_k \ge 0$ for all k. Letting $r = 2$ and $d = 1$ in Proposition 2.6 gives

$$\mathcal{N}_{[-1,1]} \le 2^n \frac{2\pi}{8} \int_{-\pi/2}^{\pi/2} \prod_{k=1}^{n} |\cos(a_k x)|\, dx.$$

Now in Proposition 3.10 let $R = \pi/2$. Then $\pi/(2R) = 1$ and by assumption there are exactly m of the a_k's with $a_k \ge 1 = \pi/(2R)$. Therefore Proposition 3.10 yields

$$\int_{-\pi/2}^{\pi/2} \prod_{k=1}^{n} |\cos(a_k x)|\, dx \le \frac{\sqrt{2\pi}}{\sqrt{(m\pi^2)/(16R^2)+\sum_{k=1}^{n-m} a_k^2}} = \frac{\sqrt{2\pi}}{\sqrt{m/4+\sum_{k=1}^{n-m} a_k^2}}.$$

Putting these estimates together gives the result. □

4.2. **Estimates when a positive proportion of the vectors have at least unit length.**

Theorem 4.2. *Let* $v_1, , \dots, v_n \in \mathbf{R}^d$ *and assume that there is a number* $\delta > 0$ *so that at least* δn *of the vectors* v_k *satisfy* $|v_k| \geq 1$*. Then if* $r \leq 2/\sqrt{d}$

$$\mathcal{N}_{Q(r)} \leq C_1(d) \frac{1}{\sqrt{\delta n}} 2^n$$

where

$$C_1(d) \leq 4(2\pi)^{d-1/2} \left(\frac{\pi}{8}\right)^d.$$

Proof. Let $v_k = (v_1^{(k)}, \dots, v_d^{(k)})$. Then if $|v_k| \geq 1$ for at least one index j the inequality $|v_j^{(k)}| \geq 1/\sqrt{d}$ holds. Therefore the hypothesis of the theorem implies that at least δn of the vectors v_k will have $|v_j^{(k)}| \geq 1/\sqrt{d}$. But then there will be a fixed j_0 so that at least $\delta n/d$ of the vectors will have $|v_{j_0}^{(k)}| \geq 1/\sqrt{d}$. By relabeling we can assume that $j_0 = 1$ and that for some $m \geq \delta n/d$ that $|v_1^{(k)}| \geq 1/\sqrt{d}$. To simplify notation let $a_k := v_1^{(k)}$ for $k = 1, \dots, m$ (so that $|a_k| \geq 1/\sqrt{d}$). Write vectors $\xi \in \mathbf{R}^d$ and $\xi = (t, \xi')$ where $t = \xi_1$ is the first coordinate of ξ and $\xi' = (\xi_1, \dots, \xi_d)$. With this notation for $1 \leq k \leq m$ we have $\langle v^k, \xi\rangle = a_k t + b_k$ where $b_k = \langle v_k', \xi'\rangle$ (so that b_k is independent of $t = \xi_1$). Therefore

$$\prod_{k=1}^n |\cos\langle v_k, \xi\rangle| \leq \prod_{k=1}^m |\cos(a_k t + b_k)|.$$

Let $Q'(2\pi/r)$ be the centered cube in $\mathbf{R}^{d-1}$ with edges of length $2\pi/r$ so that $Q'(2\pi/r) = [-\pi/r, \pi/r] \times Q'(2\pi/r)$. Using this estimate by integrating one coordinate at a time

$$\begin{aligned}
\int_{Q(2\pi/r)} \prod_{k=1}^n |\cos\langle v_k, \xi\rangle|\, d\xi &\leq \int_{Q'(2\pi/r)} \int_{-\pi/r}^{\pi/r} \prod_{k=1}^m |\cos(a_k t + b_k)|\, dt\, d\xi' \\
&= \mathrm{Vol}(Q'(2\pi/r)) \int_{-\pi/r}^{\pi/r} \prod_{k=1}^m |\cos(a_k t + b_k)|\, dt \\
&= \left(\frac{2\pi}{r}\right)^{d-1} \int_{-\pi/r}^{\pi/r} \prod_{k=1}^m |\cos(a_k t + b_k)|\, dt.
\end{aligned}$$

If $R = \pi/r$ then $\pi/(2R) = r/2 \leq 1/\sqrt{d} \leq |a_k|$ and so by Proposition 3.10 the estimate

$$\int_{-\pi/r}^{\pi/r} \prod_{k=1}^m |\cos(a_k t + b_k)|\, dt \leq \frac{8R}{\sqrt{2\pi m}} = \frac{4\sqrt{2\pi}}{\sqrt{m}r} \leq \frac{4\sqrt{2\pi}}{\sqrt{\delta n}r}$$

holds (and we have used $m \geq \delta n$). Combining with the above we have

$$\int_{Q(2\pi/r)} \prod_{k=1}^n |\cos\langle v_k, \xi\rangle|\, d\xi \leq \frac{4(2\pi)^{d-1/2}}{\sqrt{\delta n}} \left(\frac{1}{r}\right)^d.$$

Combining this with Proposition 2.6 completes the proof. □

4.3. **Estimates when the many of the vectors are not all close to a hyperplane.** The following is a very slight generalization of Theorem 1 of Halász's paper [2].

Theorem 4.3. *Let* $v_1, \dots, v_n \in \mathbf{R}^d$ *and assume that for any unit vector* u *that there are at least* m *indices* k *so that* $|\langle v_k, u\rangle| \geq 1$*. Then*

$$\mathcal{N}_{Q(2\sqrt{d})} \leq C_2(d) \left(\frac{1}{\sqrt{m}} \right)^d 2^n.$$

where

$$C_2(d) \leq \left(\frac{\pi^2 d}{\sqrt{2\pi}} \right)^d.$$

In particular this implies that if for some $\delta \in (0,1)$ *that* $m \geq \delta n$ *then*

$$\mathcal{N}_{Q(2\sqrt{d})} \leq C_2(d) \left(\frac{1}{\sqrt{\delta n}} \right)^d 2^n.$$

Proof. Let $r = 2\sqrt{d}$ in Proposition 2.6 to get

$$\mathcal{N}_{Q(2\sqrt{d})} \leq 2^n \left(\frac{\pi\sqrt{d}}{4} \right)^d \int_{Q(\pi/\sqrt{d})} \prod_{k=1}^n |\cos\langle v_k, \xi\rangle| \, d\xi.$$

Let $2R = \pi/\sqrt{d}$ om Proposition 3.9 to get

$$\int_{Q(\pi/\sqrt{d})} \prod_{k=1}^n |\cos\langle v_k, \xi\rangle| \, d\xi \leq \left(\frac{4\sqrt{d}\pi}{\sqrt{2\pi}} \right)^d \left(\frac{1}{\sqrt{m}} \right)^d.$$

Putting these two estimates together completes the proof. □

References

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