A QUESTION WHOSE ANSWER IS 42.

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Let $\mathcal{P}$ be a spherical planet of uniform density and mass M. A straight tunnel is cut through $\mathcal{P}$, but not necessary though the plant's center. Arthur Dent starts at rest at one end of the tunnel and slides, without friction or air resistance and only under the force of gravity, until he reaches the tunnel's other end.

Question: How long does the trip take?

Let C be the center of the tunnel and r the distance of C from the center of $\mathcal{P}$. Let t be the time since Arthur started his slide and $x = x(t)$ the signed distance of Arthur from C at time t. (Choose the sign so that x is positive when he is on the same side of C he started on, and negative when he is on the other side.) Let $\rho(t)$ be Arthur's distance from the center of $\mathcal{P}$. (See Figure 1.)

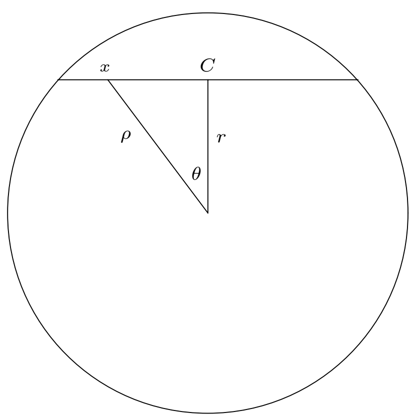

FIGURE 1

To compute the force on Arthur we need Newton's Universal Law of Gravitation: The gravitational attraction between two particles, one of mass m_1 and the other of mass m_2, is given

$$\text{force of attraction} = \frac{-Gm_1m_2}{(\text{distance between the particles})^2},$$

where G is the ***universal gravitational constant*** and the force is directed along the segment connecting the particles. We also require two non-travail facts, both special cases of more general results due to Newton (see Volume 3 of his book [2]).

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Theorem 1. *The force of gravitational attraction of a spherical planet of uniform density and mass M on an object above the surface of the planet is the same as that of a particle of mass M at the center of the planet. (That is the attraction of the planet is the same as if all its mass were at its center.)* $\square$

Theorem 2. *If $\mathcal{A}$ is the region between two spheres with a common center, and this region has a constant density, then the gravitational attraction of $\mathcal{A}$ on an object inside of the smaller sphere is zero (see Figure 2).* $\square$

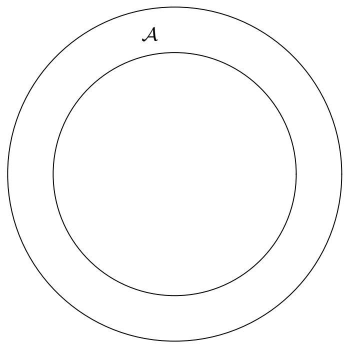

FIGURE 2. In a symmetrically hollowed out planet of uniform density there is no gravitational force on particles on the inside.

Thus, at time t, Arthur is attracted by the part of $\mathcal{P}$ that enclosed by a sphere of radius $\rho(t)$ about the center of $\mathcal{P}$, but by Theorem 2 the part of $\mathcal{P}$ at a distance greater than $\rho(t)$ does not attract him. As the density of $\mathcal{P}$ is uniform, the mass of the part of $\mathcal{P}$ inside a sphere of radius $\rho(t)$ is

$$M(t) := \left(\frac{\rho(t)}{R}\right)^3 M.$$

So the total force on Arthur is

$$\frac{-GmM(t)}{\rho(t)^2}$$

where m is Arthur's mass and this is directed toward the center of $\mathcal{P}$. However, only the component of force directed along the tunnel will contribute to his motion. If $\theta = \theta(t)$ is the angle as pictured in Figure 1 this component is the product

$$\text{Force in direction of motion} = \sin(\theta(t))\left(\frac{-GmM(t)}{\rho(t)^2}\right). \tag{1}$$

But

$$\sin(\theta(t)) = \frac{x(t)}{\rho(t)}.$$

Using this and the formula for $M(t)$ in (1) gives

$$\text{Force in direction of motion} = \frac{x(t)}{\rho(t)}\left(\frac{-Gm}{\rho(t)^2}\right)\left(\frac{\rho(t)}{R}\right)^3 M = -m\left(\frac{GM}{R^3}\right)x(t).$$

Newton's Second Law of Motion is

$$\text{mass} \times \text{acceleration} = \text{Force}.$$

If we denote the derivatives with respect to t by dots, then Arthur's acceleration is $\ddot{x}(t)$. Thus Newton's Second Law combines with our formula for the force in the direction of motion to give

$$m\ddot{x}(t) = -m\left(\frac{GM}{R^3}\right)x(t).$$

The mass m cancels:

$$\ddot{x}(t) = -\left(\frac{GM}{R^3}\right)x(t). \tag{2}$$

We rewrite this equation in terms of physical constants that are easier to measure. If a particle of mass m_o is above the surface of $\mathcal{P}$ and y is its distance from the center of $\mathcal{P}$, then Theorem 1, the Law of Gravitation, and Newton's second law combine to give

$$m_o\ddot{y}(t) = \frac{-Gm_oM}{y(t)^2}.$$

At the surface of $\mathcal{P}$, that is when $y(t) = R$, this gives the acceleration as

$$\ddot{y} = \frac{-GM}{R^2} = -g$$

where this defines g. Then g is the ***acceleration due to gravity*** at the surface of $\mathcal{P}$. Using $GM/R^2 = g$ in (2) gives

$$\ddot{x}(t) = -\left(\frac{g}{R}\right)x(t).$$

The solution of this that starts from rest at $t = 0$ is

$$x(t) = x(0)\cos(\omega t) \qquad \text{where} \qquad \omega = \sqrt{\frac{g}{R}}\,.$$

Thus the motion is simple harmonic motion where the frequency, ω, only depends on the planet's acceleration due to gravity, g, and its radius, R.

The times when he reaches the other end of the tunnel are when $\cos(\omega t) = -1$ and the smallest $t > 0$ where this occurs is

$$t_{\text{first exit}} = \frac{\pi}{\omega} = \pi\sqrt{\frac{R}{g}}\,.$$

Remarkably this is independent of the position of the tunnel in the planet, that is independent of r, but only depends on g and R.

We now specialize to when $\mathcal{P}$ is the planet Earth where

$$g \approx 9.80665\,\frac{\text{meters}}{(\text{sec})^2} \qquad \text{and} \qquad R \approx 6{,}356{,}750 \text{ meters}.$$

Using this in the formula for $t_{\text{first exit}}$ gives

$$t_{\text{first exit}} \approx 2{,}529.339 \text{ sec} \approx 42.15566 \text{ minutes}.$$

Therefore the Earth was designed so that the tunnel slide time between Atlanta and Charleston is the same as same the tunnel slide time between Los Angeles and Paris, which is the same as the tunnel slide time between the North and South poles or any other two points on the surface of the planet. And all these times are equal to 42 minutes. For a history of how this most likely came about see the book [1].

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References

1. Douglas Adams, *The Ultimate Hitchhiker's Guide to the Galaxy*, Del Rey, April 30, 2002.
2. I. S. Newton, *Philosophiae naturalis principia mathematica*, James Macklehose, Glasgow, publisher to the Univeristy, 1687.

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