Applications of Tychonoff’s Theorem.

This set of notes and problems is to show some applications of the Tychonoff product theorem. In the cases here we will have a set $A$ be looking at $[a, b]^A$, that is the set of all functions form $A$ to the interval $[a, b]$. This is the same as the product $\prod_{\alpha \in A} X_\alpha$ where $X_\alpha = [a, b]$ for all $\alpha$. The product topology on $[a, b]^A$ has a base the sets of the form

$$U(a_1, t_1, \ldots, a_n, t_n; \varepsilon) := \{f \in [a, b]^A: |f(a_i) - t_i| < \varepsilon \text{ for } i = 1, \ldots, n\}$$

where $\{a_1, \ldots, a_n\}$ is a finite subset of $A$, $t_1, \ldots, t_n \in [a, b]$ and $\varepsilon > 0$. Tychonoff’s theorem tells use that this topology on $[a, b]^A$ is compact and therefore if $F$ is a closed subset of $[a, b]^A$ a closed subset of $[a, b]^A$, then $F$ is also compact. Here are couple of examples of this type.

**Definition 1.** A normed vector space is a vector space $X$ over the field of real numbers $\mathbb{R}$ and along with a norm, which is a function $\| \cdot \| : X \rightarrow \mathbb{R}$ such that

1. $\| x \| \geq 0$ with $\| x \| = 0$ if and only if $x = 0$,
2. $\| x + y \| \leq \| x \| + \| y \|$ (i.e. the triangle inequality holds), and
3. for all $c \in \mathbb{R}$ and $x \in X$ we have $\|cx\| = |c|\|x\|$.

**Definition 2.** A bounded linear functional on the normed linear space $X$ is a linear function $f : X \rightarrow \mathbb{R}$ such that for some $C \geq 0$

$$|f(x)| \leq C\|x\|$$

holds for all $x \in X$. Let $X^*$ be the set of all bounded linear functionals $f : X \rightarrow \mathbb{R}$.

We now let

$$B := \{x \in X : \|x\| \leq 1\}.$$ 

Let

$$B^* := \{f|_B : f \in X^* \text{, and for all } x \in X \ |f(x)| \leq \|x\|\}.$$

The inequality $|f(x)| \leq \|x\|$ implies that if $x \in B$, then $|f(x)| \leq \|x\| \leq 1$. Therefore $B^*$ is a set of functions that map $B$ into $[-1, 1]$. That is $B^*$ is a subset of $[-1, 1]^B$.

**Theorem 3.** For any normed linear space $X$, the set $B^*$ is a closed subset of $[-1, 1]^B$ and therefore $B^*$ is compact with the topology it gets as a subspace of $[-1, 1]^B$. (In functional analysis this topology is called the weak* topology.)

**Problem 1.** Prove this.

**Definition 4.** A normed algebra is a normed linear space $A$ with norm $\| \cdot \|$ such that $A$ is has a product $(x, y) \mapsto xy$ that makes $A$ into an associative algebra and such that $\|xy\| \leq \|x\|\|y\|$. A multiplicative linear functional on $A$ is a linear function $f : A \rightarrow \mathbb{R}$ such that $f(xy) = f(x)f(y)$ and $|f(x)| \leq \|x\|$.
Let $B := \{ x \in A : \| x \| \leq 1 \}$ be the unit ball of $A$ and let $\Delta$ be the set of the restrictions of multiplicative linear functionals to $B$. That is

$$\Delta := \{ f|_B : f \text{ is a multiplicative linear functional} \}.$$ 

This is a subset of $[-1,1]^B$.

**Theorem 5.** For any normed algebra the set $\Delta$ is a closed subset of $[-1,1]^B$ and therefore compact.

**Problem 2.** Prove this.