Mathematics 700 Test #1 Name:______________________________

Show your work to get credit. An answer with no work will not get credit.

(1) (15 Points) Define the following:

(a) ***Linear independence***.

(b) The ***kernel*** of a linear map $S\colon V \to W$ where V and W are vector spaces.

(c) The ***rank*** of a linear map.

(d) The ***dimension*** of a vector space.

(2) (10 Points) Find (no proof required) a linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ so that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}, \qquad T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}, \qquad T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}.$$

(3) (10 Points) Find (no proof required) a basis for the set of the space of vectors $(x, y, z, w) \in \mathbf{R}^5$ that satisfy

$$\begin{aligned} x + \ \ y + \ \ z + \ \ u + \ \ v &= 0 \\ x + \ \ y + 2z + 2u + 2v &= 0 \\ x + \ \ y + \ \ z + \ \ u + 2v &= 0. \end{aligned}$$

(4) (20 Points) Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most 3. Define $T\colon \mathcal{P}_3 \to \mathcal{P}_3$ by

$$Tp(x) = p'(x+1) - p'(x).$$

(a) Find the matrix of T in the basis $\mathcal{B} := \{1, x, x^2, x^3\}$

(b) What are the rank, nullity and trace of T?

rank= ____________________

nullity= ____________________

trace= ____________________

(5) (15 Points) Show that if v_1, v_2, v_3 are vectors in the vector space $\mathbf{R}^4$ such that

$$v_1 - 4v_2 + 3v_3 = 0$$

then

$$\mathrm{Span}\{v_1, v_2\} = \mathrm{Span}\{v_1, v_2, v_3\}.$$

(6) (15 Points) Let $S\colon \mathbf{R}^5 \to \mathbf{R}^{17}$ be a linear map with $\text{nullity}(S) = 2$. Then show directly (that is without using the rank plus nullity theorem) that $\text{rank}(S) = 3$.

(7) (15 Points) Let $S\colon \mathbf{R}^{10} \to \mathbf{R}^3$ and $T\colon \mathbf{R}^{10} \to \mathbf{R}^5$ be linear maps. Then show there are two linearly independent vectors $u, v \in \mathbf{R}^{10}$ with $Su = Sv = 0$ and $Tu = Tv = 0$.