(1) (15 Points) Define the following:
(a) *Linear independence*.

(b) The *kernel* of a linear map $S: V \to W$ where $V$ and $W$ are vector spaces.

(c) The *rank* of a linear map.

(d) The *dimension* of a vector space.

(2) (10 Points) Find (no proof required) a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ so that

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$
(3) (10 Points) Find (no proof required) a basis for the set of the space of vectors \((x, y, z, w) \in \mathbb{R}^5\) that satisfy
\[
\begin{align*}
    x + y + z + u + v &= 0 \\
    x + y + 2z + 2u + 2v &= 0 \\
    x + y + z + u + 2v &= 0.
\end{align*}
\]

(4) (20 Points) Let \(P_3\) be the vector space of polynomials of degree at most 3. Define \(T: P_3 \rightarrow P_3\) by
\[
Tp(x) = p'(x + 1) - p'(x).
\]
(a) Find the matrix of \(T\) in the basis \(B := \{1, x, x^2, x^3\}\).

(b) What are the rank, nullity and trace of \(T\)?

\[
\begin{align*}
    \text{rank} &= \text{ } \\
    \text{nullity} &= \text{ } \\
    \text{trace} &= \text{ }
\end{align*}
\]
(5) (15 Points) Show that if $v_1, v_2, v_3$ are vectors in the vector space $\mathbb{R}^4$ such that

$$v_1 - 4v_2 + 3v_3 = 0$$

then

$$\text{Span}\{v_1, v_2\} = \text{Span}\{v_1, v_2, v_3\}.$$
(6) (15 Points) Let $S : \mathbb{R}^5 \to \mathbb{R}^{17}$ be a linear map with $\text{nullity}(S) = 2$. Then show directly (that is without using the rank plus nullity theorem) that $\text{rank}(S) = 3$. 
(7) (15 Points) Let $S: \mathbb{R}^{10} \to \mathbb{R}^3$ and $T: \mathbb{R}^{10} \to \mathbb{R}^5$ be linear maps. Then show there are two linearly independent vectors $u, v \in \mathbb{R}^{10}$ with $Su = Sv = 0$ and $Tu = Tv = 0$. 