(1) (15 points) Define the following:

(a) A linear map \( T : V \to V \) is \textit{diagonalizable} (where \( V \) is a finite dimensional vector space). There is a basis \( v_1, \ldots, v_n \) of \( V \) consisting of eigenvectors of \( T \). \textbf{Or}: There is a basis \( \mathcal{V} = \{v_1, \ldots, v_n\} \) of \( V \) such that the matrix \([T]_{\mathcal{V}}\) is a diagonal matrix.

(b) The \textit{adjoint} of a linear map \( S : V \to W \) between finite dimensional vector spaces \( V \) and \( W \).

The adjoint is the linear map \( S^* : W^* \to V^* \) given by

\[
S^* g = g \circ S.
\]

(Here \( \circ \) is function composition so that if \( v \in V \) and \( g \in W^* \) then \( S^* g = g(Sv) \).)

(c) \textit{Eigenvalues} and \textit{eigenvectors} of a linear map. (Be sure to be precise about the range and domain).

Eigenvalues and vectors are only defined for linear maps \( T : V \to V \) (that is the range and domain are the same). The scalar \( \lambda \in F \) is an eigenvalue iff there is a non-zero vector \( v \in V \) such that \( Tv = \lambda v \). The vector \( v \in V \) is an eigenvector iff it is not the zero vector and \( Tv = \lambda v \) for some scalar \( \lambda \in F \).

(d) The \textit{determinant} of a linear operator \( T : V \to V \) on a vector space.

Let \( \mathcal{V} = \{v_1, \ldots, v_n\} \) be any basis of \( V \) and \([T]_{\mathcal{V}}\) the matrix of \( T \) with respect to this basis. Then

\[
\det T = \det[T]_{\mathcal{V}}.
\]

(This is independent of the choice of the basis \( \mathcal{V} \) if \( V \).)

(e) \( S^\perp \) where \( S \) is a non-empty subset of a finite dimensional vector space \( V \).

\[
S^\perp = \{f \in V^* : f(x) = 0 \text{ for all } f \in V^*\}.
\]

That is \( W^\perp \) is the set of linear functional that vanish on all elements of \( S \).

(2) (10 points) Find the basis of \( \mathbb{R}^{2*} \) dual to the basis

\[
v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}
\]

\textbf{Solution:} We know that the basis of \( \mathbb{F}^{n*} \) dual to a basis \( v_1, \ldots, v_n \) of \( \mathbb{F}^n \) is made up of the rows of the inverse of the matrix \([v_1, v_2, \ldots, v_n]\) with columns \( v_1, \ldots, v_n \). In the present case

\[
[v_1, v_2] = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}.
\]

The inverse is

\[
[v_1, v_2]^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.
\]

Therefore the dual basis to \( v_1 \) and \( v_2 \) is

\[
f_1 = [-5, 3], \quad f_2 = [2, -1].
\]

Or in functional notation

\[
f_1 \begin{bmatrix} x \\ y \end{bmatrix} = -5x + 3y, \quad f_2 \begin{bmatrix} x \\ y \end{bmatrix} = 2x - y.
\]
(3) (15 points) Let $\mathcal{P}_2$ be the polynomials of degree $\leq 2$ over the real numbers and define a linear map $T: \mathcal{P}_2 \to \mathcal{P}_2$ by

$$Tp(x) = p(3x + 2).$$

Find the eigenvectors and values of $T$.

**Solution:** First we find the matrix of $T$ in some basis of $\mathcal{P}_2$. The natural choice is the basis $B := \{1, x, x^2\}$. In this basis

$$T 1 = 1 \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ T x = 3x + 2 \sim \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \ T x^2 = (3x + 2)^2 = 9x^2 + 12x + 4 \sim \begin{bmatrix} 4 \\ 12 \\ 9 \end{bmatrix}.$$

The matrix $A$ of $T$ in this basis has these vectors as columns. That is

$$A := [T]_B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix}.$$  

Using that the determinant of an upper triangular matrix is the product of the diagonal elements we see that the characteristic polynomial of $T$ is

$$\text{char}_T(x) = \det (xI - A) = (x - 1)(x - 3)(x - 9).$$

and that its roots are 1, 3 and 9. Thus the eigenvalues of $T$ (which are the same as the eigenvalues of $A$) are $\lambda = 1, 3, 9$. We now find the eigenvectors of $A$ corresponding to these eigenvalues.

For $\lambda = 1$ we want a non-zero vector in the kernel of $I - A = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 12 \\ 0 & 0 & 8 \end{bmatrix}$. This leads to the system for $x, y, z$

$$2y + 4z = 0, \ 2y + 12z = 0, \ 8z = 0.$$  

We want any non-zero solution and $x = 1, y = z = 0$ works. Thus we get the eigenvector

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 1$.

For $\lambda = 3$ we want a non-zero vector in the kernel of $I - 3A = \begin{bmatrix} -2 & 2 & 4 \\ 0 & 0 & 12 \\ 0 & 0 & 6 \end{bmatrix}$. This leads to the system

$$-2x + 2y + 4z = 0, \ 12z = 0, \ 4x = 0.$$  

A non-zero solution to this is $x = y = 1, z = 0$, which gives the eigenvector

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 3$.

For $\lambda = 9$ we want a non-zero vector in the kernel of $I - 9A = \begin{bmatrix} -8 & 2 & 4 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \end{bmatrix}$. This leads to the system

$$-8x + 2y + 4z = 0, \ -6x + 12y = 0, \ 0 = 0.$$  

A non-zero solution is $x = 1, y = 2, z = 1$, which gives the eigenvector

$$v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$  

However we are not done. This gives the eigenvectors of the matrix $A$, but we are looking for the eigenvectors of the linear operator $T$. 

For $\lambda = 1$ the element for $\mathcal{P}_3$ corresponding to $v_1$ is $p_1 = 1$, for $\lambda = 3$ the element of $\mathcal{P}_3$ corresponding to $v_2$ is $1 + x$ and for $\lambda = 9$ the element corresponding to $v_3$ is $1 + 2x + x^2 = (1 + x)^2$. 

In summary the eigenvalues of $T$ are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 9$ with corresponding eigenvectors 

$$
p_1(x) = 1, \quad p_2(x) = 1 + x, \quad p_3(x) = 1 + 2x + x^2 = (1 + x)^2.
$$

**CHECK:** We just compute 

$$
Tp_1(x) = p_1(3x + 2) = 1 = 1p_1(x),
$$  

$$
Tp_2(x) = p_2(3x + 2) = 3x + 2 + 1 = 3(x + 1) = 3p_2(x),
$$  

$$
Tp_3(x) = p_3(3x + 2) = (3x + 2 + 1)^2 = (3(x + 1))^2 = 9(x + 1)^2 = 9p_2(x).
$$

(4) (10 points) Show directly from the definitions that a linear map $T: V \rightarrow W$ between finite dimensional vector spaces is injective if and only it its adjoint $T^*: W^* \rightarrow V^*$ is surjective.

**Solution 1:** First assume that $T^*$ is surjective. Then to show $T$ is injective it is enough to show that $\ker T = \{0\}$. That is it is enough to show that $Tv = 0$ implies $v = 0$. Let $f \in V^*$. As $T^*$ is surjective there is a $g \in W^*$ with $f = T^*g$. Therefore 

$$
\langle f, v \rangle = \langle T^*g, v \rangle = \langle Tv, g \rangle = \langle 0, g \rangle = 0.
$$

This holds for all $f \in V^*$ so $v = 0$. This show $\ker T = \{0\}$ and thus that $T$ is injective.

Conversely assume that $T$ is injective. This implies that $\ker T = \{0\}$. If $\text{Image} T^*$ is not all of $V^*$ then, $\text{Image} T^*$ is a proper subspace of $V^*$ and so there is some $f_0 \notin \text{Image} T^*$. But there we can separate $f_0$ from the subspace $\text{Image} T^*$ by an evaluation. That is there is a $v_0 \in V$ with $\langle v_0, f_0 \rangle = f_0(v_0) = 1$ and $\langle v_0, f_0 \rangle = f_0(v_0) = 0$ for all $f \in \text{Image} T^*$. If $g \in W^*$ then $T^*g \in \text{Image} T^*$ and thus 

$$
0 = \langle v_0, T^*g \rangle = \langle Tv_0, g \rangle.
$$

Thus $\langle Tv_0, g \rangle = 0$ for all $g \in W^*$. This implies $Tv_0 = 0$. But $\langle v_0, f_0 \rangle = 1$ implies that $v_0 \neq 0$. This contradicts that $\ker T = \{0\}$. Thus $T^*$ is surjective. 

**Solution 2:** (This is due to Wally, though others had an equivalent but slightly less elegant version of the some proof.)

Let $\dim V = n$. Then 

$$T \text{ is injective} \iff \text{nullity}(T) = 0 \iff \text{rank}(T) = n \iff \text{rank}(T^*) = n \iff \dim \text{Image}(T^*) = n \iff \text{Image}(T^*) = V^* \iff \dim V^* = \dim V = n \iff T^* \text{ is surjective}.
$$

**Remark:** What I had in mind with the phrase “directly from the definitions” was that I did not want you to use that $\ker(T) = \text{Image} T^*$ which makes the proof a one liner (as $\ker(T) = \text{Image} (T^*) = V^*$ if and only if $\ker(T) = \{0\}$). I should have been more explicit about what could and could not be used.

(5) (10 points) Show that if a linear operator $T: V \rightarrow V$ has eigenvectors $v_1, v_2, v_3$ with distinct eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, then $v_1, v_2, v_3$ are linearly independent.

**Solution:** Let $c_1, c_2, c_3 \in \mathbb{F}$ be scalars such that 

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.
$$

As $v_1, v_2, v_3$ are eigenvectors we have 

$$Tv_1 = \lambda_1 v_1, \quad Tv_2 = \lambda_2 v_2, \quad Tv_3 = \lambda_3 v_3.
$$

Now apply $T$ to (1) and use (2) 

$$0 = T0 = c_1Tv_1 + c_2Tv_2 + c_3Tv_3 = c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + c_3\lambda_3 v_3.
$$
Multiply (1) by $\lambda_2$.

(4) \[ c_1\lambda_3 v_1 + c_2\lambda_3 v_2 + c_3\lambda_3 v_3 = 0 \]

If (4) is subtracted from (3) the $v_3$ term cancels out and we are left with

(5) \[ c_1(\lambda_3 - \lambda_1)v_1 + c_2(\lambda_3 - \lambda_2)v_2 = 0. \]

Now do the same trick again. Applying $T$ to both sides of (5) and using (2) gives

\[ c_1(\lambda_3 - \lambda_1)\lambda_1 v_1 + c_2(\lambda_3 - \lambda_2)\lambda_2 v_2 = 0. \]

Multiplying (5) by $\lambda_2$ gives

\[ c_1(\lambda_3 - \lambda_1)\lambda_2 v_1 + c_2(\lambda_3 - \lambda_2)\lambda_2 v_2 = 0. \]

Subtracting these gives

\[ c_1(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)v_1 = 0. \]

As $v_1 \neq 0$ (as it is an eigenvector) and $\lambda_1, \lambda_2, \lambda_3$ are distinct this implies $c_1 = 0$. Using $c_1 = 0$ in (5) gives

\[ c_2(\lambda_3 - \lambda_2)v_2 = 0. \]

This implies $c_2 = 0$. Now using $c_1 = c_2 = 0$ in (1) implies $c_3 = 0$. Thus we have shown that whenever (1) holds that $c_1 = c_2 = c_3 = 0$. Therefore $v_1, v_2, v_3$ are linearly independent. \( \text{done.} \)

(6) (10 points) Let $V$ be a finite dimensional vector space and $W$ a subspace of $V$ and let $v \in V$ with $v \notin W$. Let $S: W \rightarrow U$ be a linear map and $u \in U$. Show that there is a linear map $T: V \rightarrow U$ that extends $S$ and with $Tv = u$.

**Solution:** Let $k = \dim W$ and $n = \dim V$. Choose a basis $v_1, v_2, \ldots, v_k$ of $W$. Then $v \notin W = \text{Span}\{v_1, \ldots, v_k\}$ implies that $\{v_1, \ldots, v_k, \{v\}\}$ is linearly independent. So let $v_{k+1} = v$ and extend the linearly independent set $v_1, \ldots, v_k, v_{k+1}$ to a basis $v_1, \ldots, v_n$ of $V$. By the basic existence theorem for linear maps there is a linear map $T: V \rightarrow U$ such that

\[ Tv_i = \begin{cases} Sv_i, & 1 \leq i \leq k; \\ u, & i = k + 1; \\ 0, & k + 2 \leq i \leq n. \end{cases} \]

Then $Tv_i = Sv_i$ for $1 \leq i \leq k$ and thus $T|_W$ and $S$ agree on a basis of $W$. Therefore $T|_W = S$ and $Tv = Tv_{k+1} = u$. \( \text{done.} \)

(7) (10 points) Let $V$ be a vector space and $P: V \rightarrow V$ a linear map with $P^2 = P$. Show that

\[ V = \ker(P) \oplus \text{Image}(P). \]

**Solution:** We need to show that $\ker(P) + \text{Image}(P) = V$ and $\ker(P) \cap \text{Image}(P) = \{0\}$. Note for any $v \in V$ that

\[ P(v - Pv) = Pv - P^2v = Pv - Pv = 0 \]

as $P^2 = P$. Thus

\[ (v - Pv) \in \ker(P) \text{ for all } v \in V. \]

Now for any $v \in V$

\[ v = (v - Pv) + Pv. \]

Clearly $Pv \in \text{Image}(P)$ and we have just shown $(v - Pv) \in \ker(V)$. Thus every element of $V$ is a sum of an element of $\ker(P)$ and an element of $\text{Image}(P)$, whence $V = \ker(P) \oplus \text{Image}(P)$.

It remains to show that $\ker(P) \cap \text{Image}(P) = \{0\}$. Let $v \in \ker(P) \cap \text{Image}(P)$. Then $Pv = 0$ as $v \in \ker(P)$. As $v \in \text{Image}(P)$ there a $v' \in V$ with $v = Pv'$. But then, using $P^2 = P$ and $Pv = 0$,

\[ v = Pv' = P^2v' = PPv' = Pv = 0. \]

Thus if $v \in \ker(P) \cap \text{Image}(P)$, then $v = 0$. Therefore $\ker(P) \cap \text{Image}(P) = \{0\}$. \( \text{done.} \)

(8) (10 points) Let $V$ be a finite dimensional vector space and $v_1, v_2, v \in V$ such that for all $f \in V^*$

\[ f(v_1) = f(v_2) = 0 \text{ implies } f(v) = 0. \]

Show that $v$ is a linear combination of $v_1$ and $v_2$.

**Solution 1:** We wish to show that $v \in \text{Span}\{v_1, v_2\}$. Assume, toward a contradiction, that $v \notin \text{Span}\{v_1, v_2\}$. Then as $\text{Span}\{v_1, v_2\}$ is a subspace of $V$ there is a linear functional $f \in V^*$
that separates $v$ from $\text{Span}\{v_1, v_2\}$. That is $f(v) = 1$, but $f(w) = 0$ for all $w \in \text{Span}\{v_1, v_2\}$. As $v_1, v_2 \in \text{Span}\{v_1, v_2\}$ we have

$$f(v_1) = f(v_2) = 0, \quad \text{but} \quad f(v) = 1.$$ 

This clearly contradicts our assumption that $f(v_1) = f(v_2) = 0$ implies $f(v) = 0$ and we are done.

**Solution 2:** The hypothesis is that $f(v) = 0$ for any $f \in V^*$ with $f(v_1) = f(v_2) = 0$. But the set of $f \in V^*$ with $f(v_1) = f(v_2) = 0$ is $\{v_1, v_2\}^\perp$. Therefore the hypothesis can be restated as $f(v) = 0$ for all $f \in \{v_1, v_2\}^\perp$. But this is just the definition of $v \in (\{v_1, v_2\}^\perp)^\circ$. So we have

$$v \in (\{v_1, v_2\})^\circ = \text{Span}\{v_1, v_2\}$$

as $(S^\perp)^\circ = \text{Span}(S)$ for any non-empty subset $S$ of $V$. done.

(9) (10 points) Let $A \in M_{3\times3}(\mathbb{R})$ be a matrix with characteristic polynomial $x^3 - x$. Then find a diagonal matrix similar to $A$.

**Solution:** (This is basically due to Mindy, and is a bit more informative than what I did.) Note that $\text{char}_A(x) = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$. Thus the eigenvalues of $A$ are $-1, 0, 1$. Let $v_1$ be an eigenvector for $\lambda = -1$, $v_2$ an eigenvector for $\lambda = 0$ and $v_3$ an eigenvector for $\lambda = 3$. As $-1, 0, 1$ are distinct, the eigenvectors $v_1, v_2, v_3$ are linearly independent. Let $P = [v_1, v_2, v_3]$ be the matrix with $v_1, v_2, v_3$ as columns. Then we have a theorem that says that

$$P^{-1}AP = \text{diag}(-1, 0, 1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This shows that $A$ is similar to $\text{diag}(-1, 0, 1)$ done.