Mathematics 700 Test #1 Name: Solution Key

Show your work to get credit. An answer with no work will not get credit.

(1) (15 Points) Define the following:

(a) ***Linear independence***. The vectors $v_1, v_2, \ldots, v_n$ in the vector space V are linearly independent iff for scalars $c_1, c_2, \ldots, c_n \in \mathbf{F}$ if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

then $c_1 = c_2 = \cdots = c_n = 0$. RESTATEMENT: The only linear combination of $v_1, v_2, \ldots, v_n$ that vanishes is the trivial linear combination.

(b) The ***kernel*** of a linear map $S\colon V \to W$ where V and W are vector spaces. The kernel of S, denoted by $\ker(S)$, is give by

$$\ker(S) := \{v \in V : Sv = 0\}.$$

(c) The ***rank*** of a linear map. If $T\colon V \to W$ is a linear map between vector space, then the rank of T is the dimension of the subspace $\operatorname{Image}(T) := \{Tv : v \in V\}$ of W. RESTATEMENT: The rank of a linear map is the dimension of it image.

(d) The ***dimension*** of a vector space. The dimension of a vector space is the number of elements in any of its bases. (This is independent of which basis is used to define it.)

(2) (10 Points) Find (no proof required) a linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ so that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}, \qquad T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}, \qquad T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}. \tag{1}$$

Solution 1: Note that

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

Thus

$$\begin{aligned}\begin{bmatrix}x\\y\\z\end{bmatrix} &= x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\left(\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\right) + z\left(\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix}\right)\\ &= (x-y)\begin{bmatrix}1\\0\\0\end{bmatrix} + (y-z)\begin{bmatrix}1\\1\\0\end{bmatrix} + z\begin{bmatrix}1\\1\\1\end{bmatrix}.\end{aligned}$$

As so by linearity and using the values of T that are given by (1)

$$\begin{aligned}T\begin{bmatrix}x\\y\\z\end{bmatrix} &= (x-y)T\begin{bmatrix}1\\0\\0\end{bmatrix} + (y-z)T\begin{bmatrix}1\\1\\0\end{bmatrix} + zT\begin{bmatrix}1\\1\\1\end{bmatrix}\\ &= (x-y)\begin{bmatrix}1\\2\end{bmatrix} + (y-z)\begin{bmatrix}3\\4\end{bmatrix} + z\begin{bmatrix}5\\6\end{bmatrix}\\ &= \begin{bmatrix}x+2y+2z\\2x+2y+2z\end{bmatrix}\end{aligned}$$

Solution 2: We look for T as being given by a matrix.

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+fz \end{bmatrix}.$$

The the conditions (1) imply

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a+b \\ d+e \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b+c \\ d+e+f \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

This leads to the system of equations

$$\begin{aligned} a &= 1 & d &= 2 \\ a+b &= 3 & d+e &= 4 \\ a+b+c &= 5 & d+e+f &= 6 \end{aligned}$$

which can easly be solved to give

$$a = 1, \quad b = 2, \quad c = 2, \quad d = 2, \quad e = 2, \quad f = 2.$$

and so

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+2z \\ 2x+2y+2z \end{bmatrix}.$$

(3) (10 Points) Find (no proof required) a basis for the set of the space of vectors $(x, y, z, u, v) \in \mathbf{R}^5$ that satisfy

$$\begin{aligned} x + \ y + \ z + \ u + \ v &= 0 \\ x + \ y + 2z + 2u + 2v &= 0 \\ x + \ y + \ z + \ u + 2v &= 0. \end{aligned}$$

Solution: Form the matrix of this system:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \end{bmatrix}.$$

By elmentary row operations this can be reduced to row echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

So the system is equivalent to

$$\begin{aligned} x + y &= 0 \\ z + u &= 0 \\ v &= 0 \end{aligned}$$

and therefore the general solution is

$$\begin{aligned} x &= -y \\ y &= \text{any real number} \\ z &= -u \\ u &= \text{any real number} \\ v &= 0 \end{aligned}$$

Therefore the elments of the soltion space to the given equations are of the form

$$(-y, y, -u, u, 0) = y(-1, 1, 0, 0, 0) + u(0, 0, -1, 1, 0).$$

Thus $\{(-1, 1, 0, 0, 0), (0, 0, -1, 1, 0)\}$ is a basis.

(4) (20 Points) Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most 3. Define $T\colon \mathcal{P}_3 \to \mathcal{P}_3$ by

$$Tp(x) = p'(x+1) - p'(x).$$

(a) Find the matrix of T in the basis $\mathcal{B} := \{1, x, x^2, x^3\}$.
Solution: The matrix of T is the matrix whose columns are the coordiate vectors of $T1$, Tx, Tx^2, Tx^3. If $p(x) = 1$ then $p'(x) = 0$ and so $Tp(x) = p(x+1) - p(x) = 0$ in this case. Likewise if $p(x) = x$ then $p'(x) = 1$ and $Tp(x) = p'(x+1) - p'(x) = 1 - 1 = 0$. If $p(x) = x^2$, then $p'(x) = 2x$ and so $Tp(x) = p'(x+1) - p'(x) = 2(x+1) - 2x = 2$. Finally if $p(x) = x^3$, then $p'(x) = 3x^2$ and $Tp(x) = p'(x+1) - p'(x) = 3(x+1)^2 - 3x^2 = 6x + 3$. Whence

$$T1 = 0 \sim \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad Tx = 0 \sim \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$Tx^2 = 2 \sim \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad Tx^3 = 6x + 3 \sim \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore the matrix of T is

$$[T]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) What are the rank, nullity and trace of T?
Solition: As the rank, nullity and trace of T are the same as the rank, nullity, and trace of its matrix we see that

$$\operatorname{rank}(T) = 2, \qquad \operatorname{nullity}(T) = 2, \qquad \operatorname{trace}(T) = 0.$$

(5) (15 Points) Show that if v_1, v_2, v_3 are vectors in the vector space $\mathbf{R}^4$ such that

$$v_1 - 4v_2 + 3v_3 = 0$$

then

$$\operatorname{Span}\{v_1, v_2\} = \operatorname{Span}\{v_1, v_2, v_3\}.$$

Solution: From $v_1 - 4v_2 + 3v_3 = 0$ we can solve for v_3 to get $v_3 = -\frac{1}{3}v_1 + \frac{4}{3}v_2$. Let $v \in \operatorname{Span}\{v_1, v_2, v_3\}$ then, by the definition of being in the span, there are scalars $c_1, c_2, c_3 \in \mathbf{R}$ so that $v = c_1v_1 + c_2v_2 + c_3v_3$. Then, using the expression we have for v_3 in terms of v_1 and v_2, we have

$$\begin{aligned} v = c_1v_1 + c_2v_2 + c_3v_3 &= c_1v_1 + c_2v_2 + c_3\left(-\frac{1}{3}v_1 + \frac{4}{3}v_2\right) \\ &= \left(c_1 - \frac{1}{3}c_3\right)v_1 + \left(c_2 + \frac{4}{3}c_3\right)v_2 \in \operatorname{Span}\{v_1, v_2\}. \end{aligned}$$

As v was any element of $\operatorname{Span}\{v_1, v_2, v_3\}$ this implies $\operatorname{Span}\{v_1, v_2, v_3\} \subseteq \operatorname{Span}\{v_1, v_2\}$. The opposite inclusion $\operatorname{Span}\{v_1, v_2\} \subseteq \operatorname{Span}\{v_1, v_2, v_3\}$ is clear. So $\operatorname{Span}\{v_1, v_2\} = \operatorname{Span}\{v_1, v_2, v_3\}$ and we are *done.*

(6) (15 Points) Let $S\colon \mathbf{R}^5 \to \mathbf{R}^{17}$ be a linear map with $\operatorname{nullity}(S) = 2$. Then show directly (that is without using the rank plus nullity theorem) that $\operatorname{rank}(S) = 3$.

Solution: As $\dim\ker(S) = \operatorname{nullity}(S) = 2$ the subspace $\ker(S)$ of V has a basis v_1, v_2 with two elments. The vector space $\mathbf{R}^5$ is 5 dimensional therefore the basis v_1, v_2 of $\ker(S)$ can be extended to a basis v_1, v_2, v_3, v_4, v_5 of $\mathbf{R}^5$ with 5 elments.

Claim: Sv_2, Sv_4, Sv_5 is a basis of $\operatorname{Image}(S)$. To see this we first need to show that this set spans $\operatorname{Image}(S)$. Let $y \in \operatorname{Image}(S)$. Then $y = Sx$ for some $x \in \mathbf{R}^5$. Thus, as $v_1, \ldots, v_5$ is a basis of $\mathbf{R}^5$, there are scalars $x_1, x_2, x_3, x_4, x_5 \in \mathbf{R}$ with

$$x = x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 + x_5v_5.$$

As $v_1, v_2 \in \ker(S)$ we then have $Sv_1 = Sv_2 = 0$ and so

$$\begin{aligned} y = Sx &= S(x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 + x_5v_5) \\ &= x_1Sv_1 + x_2Sv_2 + x_3Sv_3 + Sx_4v_4 + x_5Sv_5 \\ &= 0 + 0 + x_3Sv_3 + Sx_4v_4 + x_5Sv_5 \\ &= x_3Sv_3 + Sx_4v_4 + x_5Sv_5 \in \operatorname{Span}\{Sv_3, Sv_4, Sv_5\}. \end{aligned}$$

Thus Sv_2, Sv_4, Sv_5 spans $\operatorname{Image}(S)$. To see that Sv_2, Sv_4, Sv_5 are linearly independent let c_3, c_4, c_5 be scalars so that

$$c_3Sv_3 + c_4Sv_4 + c_5Sv_5 = 0. \tag{2}$$

Then, by the linearity of S,

$$S(c_3v_3 + c_4v_4 + c_5v_5) = 0$$

This imples $c_3v_3 + c_4v_4 + c_5v_5 \in \ker(S)$ and v_1, v_2 are a basis for $\ker(S)$ so there are scalars $a_1, a_2 \in \mathbf{R}$ so that

$$c_3v_3 + c_4v_4 + c_5v_5 = a_1v_1 + a_2v_2.$$

Rewriting as

$$-a_1v_1 - a_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 = a_1v_1 + a_2v_2 = 0$$

and using that $v_1, v_2, \ldots, v_5$ are linearly independent this implies $c_3 = c_4 = c_5 = 0$. Thus (2) implies that $c_3 = c_4 = c_5 = 0$. Thus shows that Sv_3, Sv_4, Sv_5 is independent and completes the proof of the claim that Sv_3, Sv_4, Sv_5 is a basis of $\operatorname{Image}(S)$.

The claim shows that $\operatorname{Image}(S)$ is three dimensional. But then, by the definition of rank, $\operatorname{rank}(S) = \dim\operatorname{Image}(S) = 3$. *done.*

(7) (15 Points) Let $S\colon \mathbf{R}^{10} \to \mathbf{R}^3$ and $T\colon \mathbf{R}^{10} \to \mathbf{R}^5$ be linear maps. Then show there are two linearly independent vectors $u, v \in \mathbf{R}^{10}$ with $Su = Sv = 0$ and $Tu = Tv = 0$.

Solution 1: As $\operatorname{Image}(S) \subseteq \mathbf{R}^3$ and $\operatorname{Image}(S) \subseteq \mathbf{R}^5$ we have $\operatorname{rank}(S) \le 3$ and $\operatorname{rank}(T) \le 5$. Thus from the rank plus nullity theorem

$$\begin{aligned} 10 = \dim \mathbf{R}^{10} = \operatorname{rank}(S) + \operatorname{nullity}(S) \le 3 + \operatorname{nullity}(S), \\ 10 = \dim \mathbf{R}^{10} = \operatorname{rank}(T) + \operatorname{nullity}(T) \le 5 + \operatorname{nullity}(T). \end{aligned}$$

These imply

$$\dim \ker(S) = \operatorname{nullity}(S) \ge 7, \quad \dim \ker(T) = \operatorname{nullity}(T) \ge 5.$$

But then, using that $\ker(S)+\ker(T)$ is a subspace of $\mathbf{R}^{10}$, so that $\dim(\ker(S)+\ker(T)) \le 10$, we have

$$\begin{aligned} \dim(\ker(S)\cap\ker(T)) + 10 &\ge \dim(\ker(S)\cap\ker(T)) + \dim(\ker(S)+\ker(T)) \\ &= \dim\ker(S) + \dim\ker(T) \\ &\ge 7 + 5 = 12 \end{aligned}$$

Substract 10 from both sides of this to get $\dim(\ker(S)\cap\ker(T)) \ge 2$. As the dimension of the intersection is ≥ 2 there are linearly independent vectors $u, v \in \ker(S)\cap\ker(T)$. But then $Su = Sv = 0$ and $Tu = Tv = 0$. *done.*

Solution 2: Define a map $L\colon \mathbf{R}^{10} \to \mathbf{R}^3 \times \mathbf{R}^5 = \mathbf{R}^8$ by

$$Lx = (Sx, Tx).$$

Then for $x_1, x_2 \in \mathbf{R}^{10}$ and $c_1, c_2 \in \mathbf{R}$

$$\begin{aligned} L(c_1x_1 + x_2x_2) &= (S(c_1x_1 + x_2x_2), T(c_1x_1 + x_2x_2)) \\ &= (c_1Sx_1 + c_2Sx_2, c_1Tx_1 + c_2Tx_2) \\ &= c_1(Sx_1, Tx_1) + c_2(Sx_2, Tx_2) \\ &= c_1Lx_1 + c_2Lx_2. \end{aligned}$$

Therefore L is linear.

As $\operatorname{Image}(L) \subseteq \mathbf{R}^8$ we have $\operatorname{rank}(L) = \dim \operatorname{Image}(L) \le 8$. Therefore by the rank plus nullity theorem

$$10 = \dim \mathbf{R}^{10} = \operatorname{nullity}(L) + \operatorname{rank}(L) \le 8.$$

This implies $\dim\ker(L) = \operatorname{nullity}(L) \ge 2$. Thus there are two linearly independent vectors $u, v \in \ker(L)$. Then

$$Lu = (Su, Tu) = (0, 0), \quad Lv = (Sv, Tv) = (0, 0).$$

and we have $Su = Sv = 0$ and $Tu = Tv = 0$. *done.*