Mathematics 700 Homework
due Wednesday, October 2

Let $A \in M_{n\times n}(\mathbb{F})$ be a square matrix. That is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

Let $b \in \mathbb{F}^n$ be a column vector

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then for

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

we wish to consider the system of equations $Ax = b$. Written out explictly this is the system of n equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

Here we use some of what we have shown about linear maps, in partictular the rank plus nullity theorem, to prove some basic results about the solutions to the system (1). Associated with the non-homogenuous system (1) is the homogenuous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned} \tag{2}$$

(1) Show that the non-homogeneous system (1) is solvable for all $b \in \mathbb{F}^n$ if only only if the homogenuous system (2) has only the trival solution $x = 0$. Hint: Let $L\colon \mathbb{F}^n \to \mathbb{F}^n$ be the linear map

defined by $Lx = Ax$. Then interpret the statement "$Ax = b$ is solvable for all b" in terms of the rank of L (which is the same as the rank of A) and the statement "$Ax = 0$ has only the trivial solution" in terms of the nullity of A.

(2) Show that the range of $Lx = Ax$ is the span of the columns of A.