Mathematics 700 Homework
due Friday September 20

The rank plus nullity theorem implies the following which is one of the great (and underrated) existence theorems in elementary mathematics.

Theorem: *Let V and W be finite dimensional vector spaces and $S\colon V \to W$ a linear map. Then*

$$\operatorname{nullity}(S) \geq \dim V - \dim W.$$

Thus if $\dim V > \dim W$ *there is a non-zero solution to* $Sv = 0$.

Proof: As $\operatorname{Image}(S)$ is a subspace of W we have

$$\operatorname{rank}(S) = \dim \operatorname{Image}(S) \leq \dim W.$$

Combining this with the rank plus nullity theorem gives

$$\dim V = \operatorname{rank}(S) + \operatorname{nullity}(S) \leq \dim W + \operatorname{nullity}(S)$$

which gives the desired inequality $\operatorname{nullity}(S) \geq \dim V - \dim W$.

If $\dim V > \dim W$ then $\dim \ker(S) = \operatorname{nullity}(S) \geq \dim V - \dim W > 0$ and so $\ker(S)$ is not the zero space $\{0\}$. Thus there is a non-zero vector $v \in \ker(S)$. That is a non-zero vector with $Sv = 0$. $\square$

Here are some examples for you to practice using this theorem and the rank plus nullity theorem to prove existence and uniqueness results.

1. It is a basic fact that if $p(x)$ is a polynomial of degree $\leq n$ over the field $\mathbb{F}$ that has $(n+1)$ roots then $p(x)$ is the zero polynomial. Use this fact to show that for any elements $z_0, \ldots, z_n, w_0, \ldots w_n \in \mathbb{F}$ with $z_0, \ldots, z_n$ distinct there is a unique polynomial $p(x)$ of degree $\leq n$ so that $p(z_i) = w_i$ for $i = 1, \ldots, n$. This is often stated loosely and a little imprecisely as: It is possible to assign (or interpolate) the values of a polynomial of degree $\leq n$ at $n+1$ points arbitrarily. (Note that we have seen one proof of this, by use of Lagrange interpolation polynomials, already.) Hint: Let $\mathcal{P}_n$ be the polynomials of degree $\leq n$ and show that the map $V\colon \mathcal{P}_n \to \mathbb{F}^{n+1}$ given by $V(p) := (p(z_0), \ldots, p(z_n))$ is linear. Then apply rank plus nullity theorem.

2. (From ordinary differential equations.) Let $C^2(\mathbb{R})$ be the vector space of all real valued functions on $\mathbb{R}$ that are twice continuously differentiable. Let a and b be real numbers and let V be the two dimensional subspace of $C^2(\mathbb{R})$ spanned by the two functions $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$. Let a_0, a_1, and a_2 be be any real numbers and define $L\colon C^2(\mathbb{R}) \to C(\mathbb{R})$ by $Ly := a_2 y'' + a_1 y' + a_0 y$. Assume that the only solution to $Ly = 0$ in V is $y = 0$. Then show that for any $f \in V$ there is a unique $y_p \in V$ so that $Ly_p = f$. COMMENT: This is basically a justification for a special case of the method of undetermined coefficients you have seen in your differential equations class. It in not much harder to justify the entire method along these same lines.

3. (From partial differential equations.) Let $\mathcal{HP}^2_n$ be the homogeneous polynomials of degree n in the two variables x and y with real coefficients. That is elements of $\mathcal{HP}^2_n$ are of the form $a_0x^n + a_1x^{n-1}y + \cdots + a_kx^{n-k}y^k + \cdots + a_ny^n$ where $a_0, \ldots, a_n$ are real numbers. A ***harmonic polynomial of degree*** n is an element h of $\mathcal{HP}^1_n$ that satisfies the partial differential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

Show that there are lots of harmonic polynomials. HINT: Let $\mathcal{H}^2_n$ be the space of all harmonic polynomials of degree n. Define a linear map $\Delta\colon \mathcal{HP}^2_n \to \mathcal{HP}^2_{n-2}$ by

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Now what are the dimensions of $\mathcal{HP}^2_n$ and $\mathcal{HP}^2_{n-2}$ and what is the null space of Δ? With just a little more work you should be able to give the exact dimension of $\mathcal{H}^2_n$.

4. (From algebraic geometry) We say that a curve in the plane $\mathbb{R}^2$ has a ***polynomial parameterization*** iff it is of the form

$$\begin{aligned} c(t) &= (A(t), B(t)) \\ &= (a_0 + a_1t + a_2t^2 + \cdots + a_mt^m, b_0 + b_1t + b_2t^2 + \cdots + b_nt^n) \end{aligned}$$

where $A(t)$ and $B(t)$ are polynomials as indicated. Then show that every curve with an polynomial parameterization is algebraic in the sense that there is a nonzero polynomial $p(x, y)$ in two variables so that

$$p(A(t), B(t)) \equiv 0.$$

HINT: Let $\mathcal{P}^2_k$ be the vector space of polynomials of degree total degree $\leq k$ in the two variables x and y. Let $\mathcal{P}^1_d$ be the polynomials of degree $\leq d$ in the variable t. Given the two polynomials $A(t)$ and $B(t)$ assume that $m \leq n$ and define a map $E\colon \mathcal{P}^2_k \to \mathcal{P}^1_{nk}$ by

$$E(f)(t) := f(A(t), B(t))$$

and show that E is linear. Now compute dimensions of $\mathcal{P}^1_{nk}$ and $\mathcal{P}^2_k$ and let k get large. Are you able to give a bound on the degree of p in terms of the degrees of $A(t)$ and $B(t)$?