Mathematics 700 Homework
Due Monday, September 16

(1) Show that if V is finite dimensional and $V = U \oplus W$ for subspaces U and W, then $\dim V = \dim U + \dim W$.

(2) Let $\mathcal{P}_3$ be the vector space of real polynomials of degree ≤ 3. Let U be the subspace of $\mathcal{P}_3$ of even polynomials (that is $p(-x) = p(x)$) and let W be the subspace of odd polynomials (that is $p(-x) = -p(x)$. Show $\mathcal{P}_3 = U \oplus W$.

(3) Let V and W be vector spaces and $S, T\colon V \to W$ linear maps. Then show
   (a) The map $S + T$ is linear.
   (b) If c is a scalar, then cT is linear.

(4) Let U, V, W be vector spaces and $S\colon U \to V$, $T\colon V \to W$ linear maps. Then show $TS\colon U \to W$ is linear. (Recall that TS is the composition $T \circ S$.)

(5) Let $T\colon V \to W$ be a linear map between finite dimensional vector spaces. Show that $\operatorname{nullity}(T) \geq \dim V - \dim W$.

(6) Let $\mathcal{P}_n$ be the vector space of real polynomials of degree $\leq n$. Compute the rank and nullity of the following
   (a) The derivative $D\colon \mathcal{P}_4 \to \mathcal{P}_4$.
   (b) The maps $S\colon \mathcal{P}_4 \to \mathcal{P}_4$ given by
$$Sp(x) = \frac{1}{2}(p(x) + p(-x)).$$
   (c) The maps $A\colon \mathcal{P}_4 \to \mathcal{P}_4$ given by
$$Ap(x) = \frac{1}{2}(p(x) - p(-x)).$$
   (d) The map $T\colon \mathcal{P}_4 \to \mathcal{P}_4$ given by
$$Tp(x) = \frac{p(x+1) - 2p(x) + p(x-1)}{x}$$
   (e) The map $V\colon \mathcal{P}_n \to \mathcal{P}_n$ given by
$$Vp(x) = \int_0^x p'(t)\,dt$$