Mathematics 700 Homework
Due Friday, September 13

(1) Find a basis for the set of vectors in $\mathbf{R}^4 = \{(x, y, z, w) : x, y, z, w \in \mathbf{R}\}$ so that the two conditions

$$x + 2y + 2z + w = 0$$
$$x + 2y + 3z + 4w = 0$$

hold.

(2) Find the change of basis matrices relating the two bases

$$\mathcal{U} := \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$
$$\mathcal{V} := \{(1, 2, 3), (0, 1, 2), (0, 0, 1)\}$$

of $\mathbf{R}^3$.

(3) Let $\mathcal{P}_3$ be the vector space of all polynomials of degree ≤ 3 with real coefficients.
  (a) Show that $\mathcal{B} := \{1, (x-1), (x-1)^2, (x-1)^3\}$ is a basis for $\mathcal{P}_3$.
  (b) Find the coordinate vectors in terms of the basis $\mathcal{B}$ for x^3 and $2x^2 + 1$ in terms of the basis $\mathcal{B}$.
  (c) Find a general formula for the coordinate vector of $p(x) \in \mathcal{P}_3$ in terms of the basis $\mathcal{B}$. Hint: Taylor's theorem.

(4) Find the change of basis matrices relating the two bases

$$\mathcal{U} := \{1, x, x^2, x^3\}$$
$$\mathcal{V} := \{1, x, x(x-1), x(x-1)(x-2)\}$$

of $\mathcal{P}_3$ (the vector space of polynomials of real of degree ≤ 3.

(5) Let $\mathcal{P}_3$ be the vector space of real polynomials of degree ≤ 3 with real coefficients. Define two subsets $\mathcal{D}$ and $\mathcal{M}$ of $\mathcal{P}_3$ by

$$\mathcal{D} := \{p(x) : p(x+2) - 2p(x+1) + p(x) = 0\}$$
$$\mathcal{M} := \{p(x) : p(2x-2) = 4p(x)\}.$$

Show that both $\mathcal{D}$ and $\mathcal{M}$ are subspaces of $\mathcal{P}_3$ and find bases for each of them.

(6) Let V and W be subspaces of $\mathbf{R}^8$ so that $\dim V = 4$ and $\dim W = 5$. Show that there is a nonzero vector in $V \cap W$.

(7) If V is a finite dimensional vector space and U a subspace of V then the ***codimension of*** W ***in*** V is defined by

$$\operatorname{codim}_V W := \dim V - \dim W.$$

When the space V is clear from context we drop the subscript of V and just write $\operatorname{codim} W$. If U and W are subspaces of V then derive formulas for $\operatorname{codim}(U \cap W)$ and $\operatorname{codim}(U + W)$. (Hint: What to you know about $\dim(U \cap W)$ and $\dim(U + W)$?)

(8) Let $M_{3\times 3}(\mathbf{R})$ be the vector space of 3×3 matrices over the real numbers. That is matrices that look like

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The ***transpose*** A^t of A is matrix obtained by interchanging rows and columns of A. That is

$$A := \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

The matrix A is ***symmetric*** iff $A^t = A$ and is ***skew symmetric*** iff $A^t = -A$.

(a) Show that $M_{3\times 3}(\mathbf{R})$ is 9 dimensional by writing down a basis.

(b) Let $M^{\mathrm{sym}}_{3\times 3}(\mathbf{R})$ be the set of symmetric matrices in $M_{3\times 3}(\mathbf{R})$. Show that $M^{\mathrm{sym}}_{3\times 3}(\mathbf{R})$ is a subspace of $M_{3\times 3}(\mathbf{R})$ and find its dimension by writing down a basis.

(c) Let $M^{\mathrm{skew}}_{3\times 3}(\mathbf{R})$ be the set of skew symmetric matrices in $M_{3\times 3}(\mathbf{R})$. Show that $M^{\mathrm{skew}}_{3\times 3}(\mathbf{R})$ is a subspace of $M_{3\times 3}(\mathbf{R})$ and find its dimension by writing down a basis.

(d) Show that any 4 dimensional subspace of $M_{3\times 3}(\mathbf{R})$ contains a non-zero symmetric matrix.

(e) Give an example of a 3 dimensional subspace of $M^{\mathrm{sym}}_{3\times 3}(\mathbf{R})$ that does not contain any non-zero symmetric matrix.

The Next Quiz.

The second quiz will be on Wednesday September 11 and will cover Section 3.7 of *Schaum's Outline* (pages 73–76, echelon matrices, row canonical form, row equivalence) Know the following:

(1) The three elementary operations E_1, E_2, and E_3 on page 74.
(2) Theorem 3.7 on page 75 (that every matrix has a unique row canonical form).
(3) Be able find the row canonical form of a matrix. (Problems 3.18, 3.30 pages 98–99 would be good practice.)