(1) Find a basis for the set of vectors in $\mathbb{R}^4 = \{(x, y, z, w) : x, y, z, w \in \mathbb{R}\}$ so that the two conditions

\[
x + 2y + 2z + w = 0
\]
\[
x + 2y + 3z + 4w = 0
\]

hold.

(2) Find the change of basis matrices relating the two bases

\[\mathcal{U} := \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}\]
\[\mathcal{V} := \{(1, 2, 3), (0, 1, 2), (0, 0, 1)\}\]

of $\mathbb{R}^3$.

(3) Let $\mathcal{P}_3$ be the vector space of all polynomials of degree $\leq 3$ with real coefficients.

(a) Show that $\mathcal{B} := \{1, (x - 1), (x - 1)^2, (x - 1)^3\}$ is a basis for $\mathcal{P}_3$.

(b) Find the coordinate vectors in terms of the basis $\mathcal{B}$ for $x^3$ and $2x^2 + 1$ in terms of the basis $\mathcal{B}$.

(c) Find a general formula for the coordinate vector of $p(x) \in \mathcal{P}_3$ in terms of the basis $\mathcal{B}$. HINT: Taylor’s theorem.

(4) Find the change of basis matrices relating the two bases

\[\mathcal{U} := \{1, x, x^2, x^3\}\]
\[\mathcal{V} := \{1, x, x(x - 1), x(x - 1)(x - 2)\}\]

of $\mathcal{P}_3$ (the vector space of polynomials of real of degree $\leq 3$).

(5) Let $\mathcal{P}_3$ be the vector space of real polynomials of degree $\leq 3$ with real coefficients. Define two subsets $\mathcal{D}$ and $\mathcal{M}$ of $\mathcal{P}_3$ by

\[\mathcal{D} := \{p(x) : p(x + 2) - 2p(x + 1) + p(x) = 0\}\]
\[\mathcal{M} := \{p(x) : p(2x - 2) = 4p(x)\}\].

Show that both $\mathcal{D}$ and $\mathcal{M}$ are subspaces of $\mathcal{P}_3$ and find bases for each of them.

(6) Let $V$ and $W$ be subspaces of $\mathbb{R}^8$ so that $\dim V = 4$ and $\dim W = 5$. Show that there is a nonzero vector in $V \cap W$.

(7) If $V$ is a finite dimensional vector space and $U$ a subspace of $V$ then the **codimension of $W$ in $V$** is defined by

\[\text{codim}_V W := \dim V - \dim W\]

When the space $V$ is clear from context we drop the subscript of $V$ and just write $\text{codim} W$. If $U$ and $W$ are subspaces of $V$ then derive formulas for $\text{codim}(U \cap W)$ and $\text{codim}(U + W)$. (HINT: What to you know about $\dim(U \cap W)$ and $\dim(U + W)$?)
Let $M_{3 \times 3}(\mathbb{R})$ be the vector space of $3 \times 3$ matrices over the real numbers. That is, matrices that look like

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$ 

The \textit{transpose} $A^t$ of $A$ is matrix obtained by interchanging rows and columns of $A$. That is

$$A := \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$ 

The matrix $A$ is \textit{symmetric} iff $A^t = A$ and is \textit{skew symmetric} iff $A^t = -A$.

(a) Show that $M_{3 \times 3}(\mathbb{R})$ is 9 dimensional by writing down a basis.
(b) Let $M^{\text{sym}}_{3 \times 3}(\mathbb{R})$ be the set of symmetric matrices in $M_{3 \times 3}(\mathbb{R})$. Show that $M^{\text{sym}}_{3 \times 3}(\mathbb{R})$ is a subspace of $M_{3 \times 3}(\mathbb{R})$ and find its dimension by writing down a basis.
(c) Let $M^{\text{skew}}_{3 \times 3}(\mathbb{R})$ be the set of skew symmetric matrices in $M_{3 \times 3}(\mathbb{R})$. Show that $M^{\text{skew}}_{3 \times 3}(\mathbb{R})$ is a subspace of $M_{3 \times 3}(\mathbb{R})$ and find its dimension by writing down a basis.
(d) Show that any 4 dimensional subspace of $M_{3 \times 3}(\mathbb{R})$ contains a non-zero symmetric matrix.
(e) Give an example of a 3 dimensional subspace of $M^{\text{sym}}_{3 \times 3}(\mathbb{R})$ that does not contain any non-zero symmetric matrix.

The Next Quiz.

The second quiz will be on Wednesday September 11 and will cover Section 3.7 of \textit{Schaum’s Outline} (pages 73–76, echelon matrices, row canonical form, row equivalence) Know the following:

(1) The three elementary operations $E_1$, $E_2$, and $E_3$ on page 74.
(2) Theorem 3.7 on page 75 (that every matrix has a unique row canonical form).
(3) Be able find the row canonical form of a matrix. (Problems 3.18, 3.30 pages 98–99 would be good practice.)