Mathematics 700 Homework
Due Friday, December 6

The following is a standard part of the theory of linear operators.

Theorem 1. *Let V be a vector space over the field $\mathbf{F}$ and let $S\colon V \to V$ be a linear operator on V. Assume that there are relatively prime polynomials $p_1(x), p_2(x), \ldots, p_k(x)$ such that if $f(x) = p_1(x)p_2(x)\cdots p_k(x)$ is the product of these polynomials, then*

$$f(S) = 0.$$

Then V is the direct sum of the kernels of the linear maps $p_i(S)$. That is

$$V = \ker(p_1(S)) \oplus \ker(p_2(S)) \oplus \cdots \oplus \ker(p_k(S)).$$

Problem 1. Prove this along the following lines.

(a) Let

$$q_i(x) = \frac{f(x)}{p_i(x)} = \prod_{j\neq i} p_j(x).$$

That is $q_i(x)$ is the polynomial so that

$$p_i(x)q_i(x) = f(x).$$

Then show that $q_1(x), q_2(x), \ldots, q_k(x)$ are relatively prime.

(b) Show that there are polynomials $h_1(x), h_2(x), \ldots, h_k(x)$ such that

$$h_1(x)q_1(x) + h_2(x)q_2(x) + \cdots + h_k(x)q_k(x) = 1.$$

(c) For $1 \leq i \leq k$ let

$$P_i = h_i(S)q_i(S).$$

Show that

$$P_1 + P_2 + \cdots + P_k = I, \quad P_i^2 = P_i, \quad i \neq j \text{ implies } P_iP_j = P_jP_i = 0.$$

(d) Thus (here you can just quote an old homework problem)

$$V = \operatorname{Image}(P_1) \oplus \operatorname{Image}(P_2) \oplus \cdots \oplus \operatorname{Image}(P_k).$$

(e) To finish show that

$$\operatorname{Image}(P_i) = \ker(p_i(S)).$$

Theorem 2. *Let V be a finite dimensional vector space and let $S\colon V \to V$ be a linear operator on V. Let $\operatorname{char}_S(x)$ be the characteristic polynomial of S. Factor $\operatorname{char}_A(x)$ into powers of primes. That is*

$$\operatorname{char}_A(x) = p_1(x)^{n_1} p_2(x)^{n_2} \ldots p_k(x)^{n_k}$$

where $p_1(x), p_2(x), \ldots, p_k(x)$ are distinct irreducible polynomials. Then show that

$$V = \ker(p_1(S)^{n_1}) \oplus \ker(p_2(S)^{n_2}) \oplus \cdots \oplus \ker(p_k(S)^{n_k}).$$

This is the ***primary decomposition*** *of V under S.*

Problem 2. Prove this. Hint: Cayley-Hamilton and the theorem above.

Here are more qualifying exam questions:

Problem 3 (January 1986)**.** Let M and N be 6×6 matrices over $\mathbf{C}$, both having minimal polynomial x^3.

(1) Prove that M and N are similar if and only if they have the same rank.
(2) Give a counterexample to show that the statement is false if 6 is replaced by 7.

HINT: Think about what the elementary divisors of M and N can be.

Problem 4 (August 1987)**.** Exhibit two real matrices with no real eigenvalues which have the same characteristic polynomial and the same minimal polynomial but are not similar.

Problem 5 (August 1990)**.** *Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be given by*

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_1, -2x_2 - x_3 - 4x_4, 4x_2 + x_3)$$

(1) *Compute the characteristic polynomial of T.*
(2) *Compute the minimal polynomial of T.*
(3) *The vector space $\mathbf{R}^4$ is the direct sum of two proper T-invariant subspaces. Exhibit a basis for one of these subspaces.*

Problem 6 (August 1990)**.** Let A and B be $n \times n$ matrices with entries on the field $\mathbf{F}$ such that $A^{n-1} \neq 0$, $B^{n-1} \neq 0$, and $A^n = B^n = 0$. Prove that A and B are similar, or show, with a counterexample, that A and B are not necessarily similar. HINT: What are the possible elementary divisors of a matrix with $A^n = 0$, and $A^{n-1} \neq 0$?

Problem 7 (August 1993)**.**
Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}$$

(1) Determine the rational canonical form of A.
(2) Determine the Jordan canonical form of A.

Problem 8 (August 1998)**.** Let V be a finite dimensional vector space and $\mathcal{L}(V)$ the set of linear operators on V. Suppose $T \in \mathcal{L}(V)$. Suppose that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

where V_i is T invariant for each $i \in \{1, \ldots, k\}$. Let $m(x)$ be the minimal polynomial of T and $m_i(x)$ the minimal polynomial of T restricted to V_i, for each $i \in \{1, \ldots, k\}$. How is $m(x)$ related to the set $\{m_1(x), \ldots, m_r(x)\}$?