Problem 1. As we did not get around to doing it in class show that if $R$ is a commutative ring, and $A \in M_{n \times n}(R)$, then $A$ has an inverse if and only if $\det(A)$ is a unit in $R$. HINT: See Theorem 4.21 on Page 53 of the typeset notes.

Problem 2. Crammer’s rule for solve systems of equations is a standard result that we have not done in class. So read §4.2.1 (Pages 43–44) of the notes and do Problems 31 and 32 on Page 44.


Here are more qualifying exam questions that I think either reinforce or extend what we have covered so far in the class.

Problem 4 (August 1984). Let $V$ be the vector space over $\mathbf{R}$ of all $n \times n$ matrices with entries from $\mathbf{R}$.

(1) Prove that $\{I, A, A^2, \ldots, A^n\}$ is linearly dependent for all $A \in V$. HINT: Cayley-Hamilton.

(2) Let $A \in V$. Prove that $A$ is invertible if and only if $I$ belongs to the span of $\{A, A^2, \ldots, A^n\}$.

Problem 5 (January, 1985). (1) (a) You may skip this part. Let $V$ and $W$ be vector spaces and let $T$ be a linear operator from $V$ into $W$. Suppose that $V$ is finite-dimensional. Prove rank($T$) + nullity($T$) = dim $V$.

(b) Let $T \in L(V, V)$, where $V$ is a finite dimensional vector space. (For a linear operator $S$ denote by $\mathcal{N}(S)$ the null space and by $\mathcal{R}(S)$ the range of $S$.)

(i) Prove there is a least natural number $k$ such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1}) = \mathcal{N}(T^{k+2}) \cdots$. Use this $k$ in the rest of this problem.

(ii) Prove that $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1}) = \mathcal{R}(T^{k+2}) \cdots$.

(iii) Prove that $\mathcal{N}(T^k) \cap \mathcal{R}(T^k) = \{0\}$.

(iv) Prove $V = \mathcal{N}(T^k) \oplus \mathcal{R}(T^k)$.

Problem 6 (August, 1985). Prove that if $A$ and $B$ are $n \times n$ matrices from : and $AB = BA$, then $A$ and $B$ have a common eigenvector.

Problem 7 (August, 1987). Compute the minimal and characteristic polynomials of the following matrix. Is it diagonalizable?

\[
\begin{bmatrix}
5 & -2 & 0 & 0 \\
6 & -2 & 0 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

Problem 8 (January, 1989). Suppose that $V$ in an $n$-dimensional vector space and $T$ is a linear map on $V$ of rank 1. Prove that $T$ is nilpotent or diagonalizable. HINT: Let $W = \ker T$, then (rank plus nullity) dim $W = n - 1$. Let $v_0$ be a vector that spans Image($T$). Then split into two case. First if $v_0 \in W$ that what can you say about $TTv$ for $v \in V$? Second it $v_0 \notin W$, then choose $v_1, v_2, \ldots, v_{n-1}$ a basis of $W$ and show $v_1v_2, \ldots, v_{n-1}, v_0$ is a basis of $V$. What is the matrix of $T$ in this basis?
Problem 9 (August, 1990). If $M$ is the $n \times n$ matrix

\[
M = \begin{bmatrix}
  x & a & a & \cdots & a \\
  a & x & a & \cdots & a \\
  a & a & x & \cdots & a \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a & a & a & \cdots & x
\end{bmatrix}
\]

then prove that $\det M = [x + (n - 1)a](x - a)^{n-1}$.

Problem 10 (August, 1991). Let $A$ be an $n \times n$ with entries from the field $F$. Suppose that $A^2 = A$. Prove that the rank of $A$ is equal to the trace of $A$. HINT: As $A^2 = A$ then we can write $F^n$ as a direct sum $F^n = \text{Image}(A) \oplus \ker(A)$. Choose a basis $v_1, \ldots, v_r$ of $\text{Image}(A)$ and a basis $v_{r+1}, \ldots, v_n$ of $\ker(A)$. Then show $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a basis of $F^n$. What is the matrix of $A$ in the basis $\mathcal{V}$?

Problem 11 (January, 1996). Let $P_3$ be the vector space of all with coefficients from $\mathbb{R}$ and of degree at most 3. Define a linear $T : P_3 \rightarrow P_3$ by $(Tf)(x) = f(2x - 6)$. Is $T$ diagonalizable? Explain why.

Problem 12 (January, 1996). Let $P_3$ be the vector space of all with coefficients from $\mathbb{R}$ and of degree at most 3. Define a linear $T : P_3 \rightarrow P_3$ by $(Tf)(x) = f(2x - 6)$. Is $T$ diagonalizable? Explain why.

Problem 13 (August, 1996). Suppose that $A$ is a $3 \times 3$ real orthogonal matrix, i.e., $A^t = A^{-1}$, with determinant $-1$. Prove that $-1$ is an eigenvalue of for $A$. HINT: Recall that $-1$ is an eigenvalue of $A$ if and only if $\det(A - (-1)I) = \det(A + I) = 0$. Now note

\[
\det(A + I) = \det((A + I)^t) = \det(A^t + I) = \det(A^{-1} + I)
\]

\[
= \det(A^{-1}(I + A)) = \det(A^{-1}) \det(I + A).
\]

Now what is $\det(A^{-1})$.

Problem 14 (January, 1998). For any nonzero scalar $a$, show that there are no real $n \times n$ matrices $A$ and $B$ such that $AB - BA = aI$. HINT: Take the trace.

Problem 15 (January, 1998). Suppose that $A$ and $B$ are diagonalizable matrices over a field $F$. Prove that they are simultaneously diagonalizable, that is there there exists an invertible matrix $P$ such that $PAP^{-1}$ and $PBP^{-1}$ are both diagonal, if and only if $AB = BA$. 