Mathematics 700 Homework
Due Monday, December 2

Problem 1. As we did not get around to doing it in class show that if R is a commutative ring, and $A \in M_{n\times n}(R)$, then A has an inverse if and only if $\det(A)$ is a unit in R. HINT: See Theorem 4.21 on Page 53 of the typeset notes. □

Problem 2. Crammer's rule for solve systems of equations is a standard result that we have not done in class. So read §4.2.1 (Pages 43–44) of the notes and do Problems 31 and 32 on Page 44. □

Problem 3. Read Pages 56–57 of the notes and do Problem 36 on Page 57. □

Here are more qualifying exam questions that I think either reinforce or extend what we have covered so far in the class.

Problem 4 (August 1984). *Let V be the vector space over $\mathbf{R}$ of all $n \times n$ matrices with entries from $\mathbf{R}$.*

(1) *Prove that $\{I, A, A^2, \dots, A^n\}$ is linearly dependent for all $A \in V$.* HINT: *Cayley-Hamilton.*
(2) *Let $A \in V$. Prove that A is invertible if and only if I belongs to the span of $\{A, A^2, \dots, A^n\}$.* □

Problem 5 (January, 1985). (1) (a) ***You may skip this part.*** Let V and W be vector spaces and let T be a linear operator from V into W. Suppose that V is finite-dimensional. Prove $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$.

(b) Let $T \in L(V,V)$, where V is a finite dimensional vector space. (For a linear operator S denote by $\mathcal{N}(S)$ the null space and by $\mathcal{R}(S)$ the range of S.)
  (i) Prove there is a least natural number k such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1}) = \mathcal{N}(T^{k+2}) \cdots$ Use this k in the rest of this problem.
  (ii) Prove that $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1}) = \mathcal{R}(T^{k+2}) \cdots$
  (iii) Prove that $\mathcal{N}(T^k) \cap \mathcal{R}(T^k) = \{0\}$.
  (iv) Prove $V = \mathcal{N}(T^k) \oplus \mathcal{R}(T^k)$. □

Problem 6 (August, 1985). Prove that if A and B are $n \times n$ matrices from : and $AB = BA$, then A and B have a common eigenvector. □

Problem 7 (August, 1987). Compute the minimal and characteristic polynomials of the following matrix. Is it diagonalizable?

$$\begin{bmatrix} 5 & -2 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Problem 8 (January, 1989). Suppose that V in an n-dimensional vector space and T is a linear map on V of rank 1. Prove that T is nilpotent or diagonalizable. HINT: Let $W = \ker T$, then (rank plus nullity) $\dim W = n - 1$. Let v_0 be a vector that spans $\operatorname{Image}(T)$. Then split into two case. First if $v_0 \in W$ that what can you say about TTv for $v \in V$? Second it $v_0 \notin W$, then choose $v_1, v_2, \dots, v_{n-1}$ a basis of W and show $v_1 v_2, \dots, v_{n-1}, v_0$ is a basis of V. What is the matrix of T in this basis?

Problem 9 (August, 1990)**.** If M is the $n \times n$ matrix

$$M = \begin{bmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x \end{bmatrix}$$

then prove that $\det M = [x + (n-1)a](x-a)^{n-1}$. $\square$

Problem 10 (August, 1991)**.** Let A be an $n \times n$ with entries from the field $\mathbf{F}$. Suppose that $A^2 = A$. Prove that the rank of A is equal to the trace of A. HINT: As $A^2 = A$ then we can write $\mathbf{F}^n$ as a direct sum $\mathbf{F}^n = \operatorname{Image}(A) \oplus \ker(A)$. Choose a basis $v_1, \dots, v_r$ of $\operatorname{Image}(A)$ and a basis $v_{r+1}, \dots, r_n$ of $\ker(A)$. Then show $\mathcal{V} = \{v_1, \dots, v_n\}$ is a basis of $\mathbf{F}^n$. What is the matrix of A in the basis $\mathcal{V}$? $\square$

Problem 11 (January, 1996)**.**
Let $\mathcal{P}_3$ be the vector space of all with coefficients from $\mathbf{R}$ and of degree at most 3. Define a linear $T : \mathcal{P}_3 \to \mathcal{P}_3$ by $(Tf)(x) = f(2x-6)$. Is T diagonalizable? Explain why. $\square$

Problem 12 (January, 1996)**.** Let $\mathcal{P}_3$ be the vector space of all with coefficients from $\mathbf{R}$ and of degree at most 3. Define a linear $T : \mathcal{P}_3 \to \mathcal{P}_3$ by $(Tf)(x) = f(2x-6)$. Is T diagonalizable? Explain why. $\square$

Problem 13 (August, 1996)**.** Suppose that A is a 3×3 real orthogonal matrix, i.e., $A^t = A^{-1}$, with determinant -1. Prove that -1 is an eigenvalue of for A. HINT: Recall that -1 is an eigenvalue of A if and only if $\det(A - (-1)I) = \det(A + I) = 0$. Now note

$$\begin{aligned} \det(A+I) &= \det\left((A+I)^t\right) = \det(A^t + I) = \det(A^{-1} + I) \\ &= \det(A^{-1}(I + A)) = \det(A^{-1}) \det(I + A). \end{aligned}$$

Now what is $\det(A^{-1})$. $\square$

Problem 14 (January, 1998)**.** For any nonzero scalar a, show that there are no real $n \times n$ matrices A and B such that $AB - BA = aI$. HINT: Take the trace. $\square$

Problem 15 (January, 1998)**.** Suppose that A and B are diagonalizable matrices over a field $\mathbb{F}$. Prove that they are simultaneously diagonalizable, that is there there exists an invertible matrix P such that PAP^{-1} and PBP^{-1} are both diagonal, if and only if $AB = BA$. $\square$