Mathematics 700 Homework
Due Friday, November 22

The following results are a important parts of the theory of linear operators on a vector space.

Proposition 1. *Let V be a finite dimensional vector space over the field $\mathbf{F}$ and let $S\colon V \to V$ be a linear operator on V. Let $\mathbf{F}[x]$ be the ring of polynomials over $\mathbf{F}$ and let I be the set*

$$I := \{f(x) \in \mathbf{F}[x] : f(S) = 0\}.$$

*Then I is a non-zero ideal in $\mathbf{F}[x]$ and there is a unique monic polynomial $\min_S(x)$ such that $I = \langle \min_S(x) \rangle$. (Here $\langle \min_S(x) \rangle$ is the principle idea generated by $\min_S$.) The polynomial $\min_S(x)$ is the **minimal polynomial** of S and is a very important invariant of S.*

Problem 1. Prove this. Hint: That $I \neq \langle 0 \rangle$ was on an old homework assignment. $\square$

Theorem 1. *Let V be a vector space over the field $\mathbf{F}$ and let $S\colon V \to V$ be a linear operator on V. Assume there are relatively prime polynomials $f(x), g(x) \in \mathbf{F}[x]$ such that*

$$f(S)g(S) = 0.$$

Then show that

$$V = \ker(f(S)) \oplus \ker(g(S)).$$

Problem 2. Prove this along the following lines.

(1) As $f(x)$, $g(x)$ are relatively prime, there are polynomials $p(x), q(x) \in \mathbf{F}[x]$ with
$$p(x)f(x) + q(x)g(x) = 1.$$

(2) With this $p(x), q(x)$ let
$$P = p(S)f(S), \qquad Q = q(S)g(S)$$
and show that
$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, PQ = QP = 0.$$

(3) Thus (here you just have to quote an old homework problem),
$$V = \operatorname{Image}(P) \oplus \operatorname{Image}(Q).$$

(4) To finish show that
$$\operatorname{Image}(P) = \ker(g(S)), \qquad \operatorname{Image}(Q) = \ker(f(S)).$$ $\square$

Problem 3. Let $S\colon V \to V$ be a linear operator on the real vector space V and assume that $S^3 + S^2 + S + I = 0$. Show that

$$V = \ker(S + I) \oplus \ker(S^2 + I).$$ $\square$

Problem 4. Let $A, B \in M_{n\times n}(\mathbf{F})$ be square matrices over a field. Assume that A and B are similar. Then show that A and B have the same characteristic polynomial and the same minimal polynomial. $\square$

Now here are some problems off of old qualifying exams.

Problem 5. Let n be a positive integer. Define

$G = \{A : \ A$ is an $n \times n$ matrix with only integer entries and $\det A \in \{-1, +1\}\}$,

$H = \{A : \ A$ is an invertible $n \times n$ matrix and both A and A^{-1} have only integer entries$\}$.

Prove $G = H$.

Problem 6. Give an example of two 4×4 matrices that are not similar but that have the same minimal polynomial. HINT: A good place to look for examples is with both matrices being diagonal projections.

Problem 7 (January 1984)**.** Let $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$, where D and G are $n \times n$ matrices. If $DF = FD$ prove that $\det A = \det(DG - FE)$. HINT: I have forgotten how to do this, but it looked like fun, so I thought it would be nice to spread the happiness around. If you don't feel in the mood to do linear algebra for fun this week, don't spent too much time on this one.