Mathematics 700 Homework
Due Monday, November 18

Definition 1. *Let V be a vector space. Then a linear operator $P\colon V \to V$ is a* ***projection*** *iff $P^2 = P$.* $\square$

You all know the proof of

Proposition 1. *If $P\colon V \to V$ is a projection, then*

$$V = \ker(P) \oplus \operatorname{Image}(P).$$

$\square$

There is a converse of sorts to this.

Proposition 2. *Let $V = U \oplus W$. Then there is a unique projection $P\colon V \to V$ with*

$$U = \ker(P), \qquad W = \operatorname{Image}(P).$$

Problem 1. Prove this. Hint: As $V = U \oplus W$ each $v \in V$ has a unique representation as $v = u + w$ with $u \in U$ and $w \in W$. Set $Pv = w$ and show this is linear and has the desired properties. Then show uniqueness. $\square$

Problem 2. Let $P\colon V \to V$ be a projection and set $Q = I - P$. Then show that $\operatorname{Image}(Q) = \ker(P)$ and $\ker(Q) = \operatorname{Image}(P)$ and

$$Q^2 = Q, \qquad P + Q = I, \quad PQ = QP = 0, \quad V = \operatorname{Image}(P) \oplus \operatorname{Image}(Q).$$

$\square$

Problem 3. Let $P, Q\colon V \to V$ be projections such that

$$P + Q = I, \quad \textit{and} \quad PQ = QP = 0.$$

Then show

$$V = \operatorname{Image}(P) \oplus \operatorname{Image}(Q).$$

Hint: No hard work required. Just reduce this to the last problem. $\square$

Definition 2. *Let V be a vector space and $W_1, \ldots, W_k$ non-zero subspaces of V. Then $W_1, \ldots, W_k$ are* ***independent*** *iff*

$$w_i \in W_i \ \textit{ for } 1 \le i \le k \textit{ and } \sum_{i=1}^{k} w_i = 0 \qquad \Longrightarrow \qquad w_i = 0 \textit{ for } 1 \le i \le k.$$

$\square$

Definition 3. *If V is a vector space and $W_1, \ldots, W_k$ non-zero sub-spaces of V, then V is the* ***direct sum*** *of $W_1, \ldots, W_k$ iff then are independent and span V. (By span we mean that $W_1 + \cdots + W_k = V$.) In this case we write*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

$\square$

Problem 4. If V is finite dimensional and $V = W_1 \oplus \cdots \oplus W_k$, then show

$$\dim(V) = \sum_{i=1}^{k} \dim W_i.$$

$\square$

Problem 5. Let V be a vector space and let $P_1, \dots, P_k \colon V \to V$ be (i.e. $P_i^2 = P_i$) such that

$$i \neq j \quad \Longrightarrow \quad P_i P_j = P_j P_i = 0$$

and

$$P_1 + P_2 + \cdots + P_k = I.$$

Show

$$V = \operatorname{Image}(P_1) \oplus \operatorname{Image}(P_2) \oplus \cdots \oplus \operatorname{Image}(P_k).$$

Problem 6. Let V be a vector space over the real numbers and let $S \colon V \to V$ be a linear operator on V. Assume that $A^2 - 5A + 6I = 0$. Show that

$$V = \ker(A - 2I) \oplus \ker(A - 3I).$$

Hint: If you are clever you can reduce this to Problem 3 above with $P = aI + bA$ and $Q = cI + dA$ for the right choice of scalars a, b, c, d.

Problem 7. Here is a determinant formula that every mathematician seems to know and love. Let

$$V(x_1, x_2, \dots, x_n) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

be the Vandermonde. Matrix Show

$$\det V(x_1, x_2, \dots, x_n) = \prod_{i<j} (x_j - x_i)$$

Hint: Use induction. Multiply each row by x_1 and substract it from the next row its the below it (this will not change the value of the determinant). Then you should find that

$$\det V(x_1, x_2, \dots, x_n) = (x_n - x_1)(x_{n-1} - x_1) \cdots (x_2 - x_1) \det V(x_2, x_3, \dots, x_n) \qquad \square$$