Mathematics 700 Homework
Due Friday, November 1

Problem 1. Let $A \in M_{m\times n}(\mathbf{F})$ with $\operatorname{rank}(A) = r$. Then show there is an invertible $m \times m$ matrix P and an invertible $n \times n$ matrix Q so that PAQ has r ones down the main diagonal and all other elements zero. HINT: Use elementary row and column matrices to put A reduce A to the required form. Or it is possible to use the methods of Problem 1 on the Homework which was due on Wednesday, October 9 to do this problem.

Problem 2. Let V and W be finite dimensional vector spaces and $T\colon V \to W$ with adjoint $T^*\colon W^* \to V^*$. Then show

$$\ker(T^*) = \operatorname{Image}(T)^\perp \qquad \text{and} \qquad \operatorname{Image}(T^*) = \ker(T)^\perp.$$

□

The next problem shows that our results

$$\ker(T) = \operatorname{Image}(T^*)^\circ, \tag{1}$$

$$\operatorname{Image}(T) = \ker(T^*)^\circ, \tag{2}$$

$$\ker(T^*) = \operatorname{Image}(T)^\perp, \tag{3}$$

$$\operatorname{Image}(T^*) = \ker(T)^\perp. \tag{4}$$

are less abstract than they seem at first.

Problem 3. Let $A \in M_{m\times n}(\mathbf{F})$. Then we wish to understand when the set of linear equations

$$Ax = b \tag{5}$$

has a solution. (Here $x \in \mathbf{F}^n$ and $b \in \mathbf{F}^m$. Thus $Ax = b$ is a system of m linear equations in n unknowns.) Show the following

(1) Solutions to (5) are unique (if they exist) if and only if $\operatorname{nullity}(A) = 0$.
(2) The equation $Ax = b$ has a solution if and only it b is orthogonal to all the solutions to $yA = 0$ where $y \in \mathbf{F}^{m*}$ = space of length m row vectors. HINT: $Ax = b$ has a solution if and only if $b \in \operatorname{Image}(A)$ so one of the equations above is relevant. Also note $A^t y^t = (yA)^t$, so $yA = 0$ if and only if $A^t y^t = 0$.
(3) $x \in \mathbf{F}^n$ is a solution to $Ax = 0$ if and only if x is orthogonal to all the row vectors yA with $y \in \mathbf{F}^{m^*}$. □

The following gives us some review on quotient spaces and the first homomorphism theorem. First recall (see homework due Due Wednesday, October 9) that we have

Theorem 1 (First Homomorphism Theorem)**.** *Let V and U be vector spaces and let $T\colon V \to U$ be a surjective (that is onto) linear map. Then the vector space $V/\ker T$ is isomorphic to U. (Written as $U \approx V/\ker T$.)* □

The is often stated as:

Theorem 2 (Restatement of First Homomorphism Theorem)**.** *Let V and W be vector spaces and let $T\colon V \to W$ be linear. Then show $V/\ker T$ is isomorphic to* $\operatorname{Image} T$. □

Problem 4. Prove this from the first form of the First Homomorphism Theorem. HINT: Let $U = \operatorname{Image} T$. □

We also know that

Proposition 1. *If V is a finite dimensional vector space and W is a subspace of V, then*

$$\dim V/W = \operatorname{codim}_V W = \dim V - \dim W.$$

□

While we used the Rank plus Nullity Theorem to prove this it is not hard to prove this directly. We can now use it to give another proof of the Rank plus Nullity theorem.

Problem 5. Let V and W be vector spaces with V finite dimensional. Let $T\colon V \to W$ be linear. Then use Proposition 1 and Theorem 2 to give a *short* proof that

$$\operatorname{rank} T = \operatorname{codim} \ker T.$$

Thus $\operatorname{rank} T = \dim V - \dim \ker T = \dim V - \operatorname{nullity} T$. □