Problem 1. Let $A \in M_{m \times n}(F)$ and $B \in M_{n \times p}$. Assume that $A$ and $B$ are partitioned into blocks:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where $A_{ij}$ is $m_i \times n_j$ and $B_{ij}$ is $n_i \times p_j$ for $1 \leq i, j \leq 2$. (Thus $m_1 + m_2 = m$, $n_1 + n_2 = n$ and $p_1 + p_2 = p$.) Show that we can multiply $A$ and $B$ “block at a time”. That is show

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

(For example in this formula $A_{12}B_{22}$ is the $m_1 \times p_2$ matrix obtained from the matrix multiplication of the $m_1 \times n_2$ matrix $A_{12}$ with the $n_2 \times p_2$ matrix $B_{22}$.)

Problem 2. State the generalization of the last problem to the block product of $AB$ (with $A \in M_{m \times n}(F)$ and $B \in M_{n \times p}$) where $A$ and $B$ are partitioned into a larger number of blocks. You do not have to prove this (as it is just like the last problem, but messier). Hint: See Schaum’s §2.12 page 41.

Problem 3. The following are some standard facts about the relationship between how a linear operator acts on a vector space and the form of its matrix representation in an appropriate basis. This a basic part of the theory and something you should make a point of remembering for the rest of your lives. In these problems let $V$ be a finite dimensional vector space and $T : V \to V$ a linear operator on $V$.

(a) Recall that a subspace $W \subseteq V$ is invariant under $T$ iff $w \in W$ implies $Tw \in W$. Let $k = \dim W$ and $n = \dim V$. Choose a basis $\mathcal{V} = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ so that $W = \text{Span}\{v_1, \ldots, v_k\}$. Then show that $W$ is invariant under $T$ if and only if the matrix $A := [T]_\mathcal{V}$ has the block upper triangular form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where $A_{11}$ is $k \times k$ and $A_{22}$ is $(n-k) \times (n-k)$. Hint: This is easy so please don’t make it hard.

(b) A basis $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a triangular basis for $T$ iff for $Tv_k \in \text{Span}\{v_1, \ldots, v_k\}$ for $k = 1, \ldots, n$. Show that the basis $\mathcal{V}$ is triangular for $T$ if and only if the matrix $A = [T]_\mathcal{V}$ is upper triangular. Thus when $n = 4$ this means that the basis $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ is triangular if and only if the matrix $A = [T]_\mathcal{V}$ is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$
(In general a matrix \(A = [a_{ij}]\) is upper triangular iff \(a_{ij} = 0\) for \(i > j\).) If you wish you can just do the proof in the case of \(n = 4\).

**Problem 4.** Let \(V\) be a finite dimensional vector space and \(U, W\) subspaces of \(V\). Then show the following:

(a) \(U \subseteq W\) if and only if \(U^\perp \supseteq W^\perp\) (note the direction of the inclusion is reversed).

(b) \(U = W\) if and only if \(U^\perp = W^\perp\).

**Problem 5.** Let \(V\) be a finite dimensional vector space and let \(f_1, f_2, f \in V^*\) be linear functionals on \(V\). Show that \(f\) is a linear combination of \(f_1\) and \(f_2\) if and only if \(\ker(f_1) \cap \ker(f_2) \subseteq \ker(f)\). **Remark:** This generalizes as follows: If \(f_1, \ldots, f_k, f \in V^*\) then \(f \in \text{Span}\{f_1, \ldots, f_k\}\) if and only if \(\bigcap_{i=1}^k \ker(f_i) \subseteq \ker(f)\). The proof for \(k = 2\) and the general case are basically the same.

**Problem 6.** Let \(V\) and \(W\) be finite dimensional vector spaces. Let \(f \in V^*\) be linear functional on \(V\) and \(w \in W\) an element of \(W\). Define a map \(w \otimes f : V \to W\) by

\[(w \otimes f)(v) = f(v)w.\]

It is not hard to see that \(w \otimes f\) is linear.

(a) Show if \(f \neq 0\) and \(w \neq 0\) then \(w \otimes f\) has rank one.

(b) Conversely show that if \(T : V \to W\) has rank one that there is an \(f \in V^*\) and a \(w \in W\) so that \(T = w \otimes f\). **Hint:** Choose \(w \in \text{Image}(T)\) with \(w \neq 0\). Then as \(T\) has rank one \(\{w\}\) is a basis of \(\text{Image}(T)\). For any \(v \in V\) we have \(Tv \in \text{Image}(T)\) and thus \(Tv = cw\) for some scalar \(c\). How does \(c\) depend on \(v\)?

**Problem 7.** Finally here is one just for fun. (Well we will also be using this result latter.) Recall from calculus that if \(a \in \mathbb{R}\) has \(|a| < 1\) then \(1/(1-a)\) can be computed by the geometric series

\[(1) \quad \frac{1}{1-a} = 1 + a + a^2 + a^3 + \cdots = \sum_{k=0}^{\infty} a^k.\]

This has a very neat generalization to matrices. A matrix \(A \in M_{n \times n}\) is **nilpotent** iff for some \(m \geq 1\) we have \(A^m = 0\). As examples you can check that all of the following are nilpotent.

\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & -4 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 4 & 9 & 0 & 0 \\ -3 & 7 & 4 & 0 \end{bmatrix}
\]

Show that if \(A\) is nilpotent, say \(A^m = 0\), then \(I - A\) is invertible and

\[(I - A)^{-1} = I + A + A^2 + \cdots + A^{m-1}.\]

Note this is exactly like putting \(a = A\) in the series (1) using that \(A^k = 0\) for \(k \geq m\). **Hint:** Set \(B = I + A + A^2 + \cdots + A^{m-1}\) and by direct multiplication show that \((I - A)B = B(I - A) = I\).