Mathematics 700 Homework
Due Monday, October 28

Problem 1. Let $A \in M_{m\times n}(\mathbf{F})$ and $B \in M_{n\times p}$. Assume that A and B are partitioned into blocks:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where A_{ij} is $m_i \times n_j$ and B_{ij} is $n_i \times p_j$ for $1 \le i, j \le 2$. (Thus $m_1 + m_2 = m$, $n_1 + n_2 = n$ and $p_1 + p_2 = p$.) Show that we can multiply A and B "block at a time". That is show

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

(For example in this formula $A_{12}B_{22}$ is the $m_1 \times p_2$ matrix obtained from the matrix multiplication of the $m_1 \times n_2$ matrix A_{12} with the $n_2 \times p_2$ matrix B_{22}.)

Problem 2. State the generalization of the last problem to the block product of AB (with $A \in M_{m\times n}(\mathbf{F})$ and $B \in M_{n\times p}$) where A and B are partitioned into a larger number of blocks. You do not have to prove this (as it is just like the last problem, but messier). HINT: See Schaum's §2.12 page 41.

Problem 3. The following are some standard facts about the relationship between how a linear operator acts on a vector space and the form of its matrix representation in an appropriate basis. This a basic part of the theory and something you should make a point of remembering for the rest of your lives. In these problems let V be a finite dimensional vector space and $T\colon V \to V$ a linear operator on V.

(a) Recall that a subspace $W \subset V$ is ***invariant under*** T iff $w \in W$ implies $Tw \in W$. Let $k = \dim W$ and $n = \dim V$. Choose a basis $\mathcal{V} = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ so that $W = \operatorname{Span}\{v_1, \ldots, v_k\}$. Then show that W is invariant under T if and only if the matrix $A := [T]_{\mathcal{V}}$ has the block upper triangular form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is $k \times k$ and A_{22} is $(n-k) \times (n-k)$. HINT: This is easy so please don't make it hard.

(b) A basis $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a ***triangular basis for*** T iff for $Tv_k \in \operatorname{Span}\{v_1, \ldots, v_k\}$ for $k = 1, \ldots, n$. Show that the basis $\mathcal{V}$ is triangular for T if and only if the matrix $A = [T]_{\mathcal{V}}$ is upper triangular. Thus when $n = 4$ this means that the basis $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ is triangular if and only if the matrix $A = [T]_{\mathcal{V}}$ is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$

(In general a matix $A = [a_{ij}]$ is upper triangular iff $a_{ij} = 0$ for $i > j$.) If you wish you can just do the proof in the case of $n = 4$.

Problem 4. Let V be a finite dimensional vector space and U, W subspaces of V. Then show the following:

(a) $U \subseteq W$ if and only if $U^\perp \supseteq W^\perp$ (note the direction of the inclusion is reversed).
(b) $U = W$ if and only if $U^\perp = W^\perp$.

Problem 5. Let V be a finite dimensional vector space and let $f_1, f_2, f \in V^*$ be linear functionals on V. Show that f is a linear combination of f_1 and f_2 if and only if $\ker(f_1) \cap \ker(f_2) \subseteq \ker(f)$. **Remark:** This generalizes as follows: If $f_1, \ldots, f_k, f \in V^*$ then $f \in \operatorname{Span}\{f_1, \ldots, f_k\}$ if and only if $\bigcap_{i=1}^k \ker(f_i) \subseteq \ker(f)$. The proof for $k = 2$ and the general case are basically the same.

Problem 6. Let V and W be finite dimensional vector spaces. Let $f \in V^*$ be linear functional on V and $w \in W$ an element of W. Define a map $w \otimes f \colon V \to W$ by

$$(w \otimes f)(v) = f(v)w.$$

It is not hard to see that $w \otimes f$ is linear.

(a) Show if $f \neq 0$ and $w \neq 0$ then $w \otimes f$ has rank one.
(b) Conversely show that if $T \colon V \to W$ has rank one that there is an $f \in V^*$ and a $w \in W$ so that $T = w \otimes f$. Hint: Choose $w \in \operatorname{Image} T$ with $w \neq 0$. Then as T has rank one $\{w\}$ is a basis of $\operatorname{Image}(T)$. For any $v \in V$ we have $Tv \in \operatorname{Image}(T)$ and thus $Tv = cw$ for some scalar c. How does c depend on v?

Problem 7. Finally here is one just for fun. (Well we will also be using this result latter.) Recall from calculus that if $a \in R$ has $|a| < 1$ then $1/(1-a)$ can be computed by the geometric series

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \cdots = \sum_{k=0}^{\infty} a^k. \tag{1}$$

This has a very neat generalization to matrices. A matrix $A \in M_{n\times n}$ is ***nilpotent*** iff for some $m \geq 1$ we have $A^m = 0$. As examples you can check that all of the following are nilpotent.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 2 & -4 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 4 & 9 & 0 & 0 \\ -3 & 7 & 4 & 0 \end{bmatrix}$$

Show that if A is nilpotent, say $A^m = 0$, then $I - A$ is invertiable and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{m-1}.$$

Note this is exactly like putting $a = A$ in the series (1) using that $A^k = 0$ for $k \geq m$. Hint: Set $B = I + A + A^2 + \cdots + A^{m-1}$ and by direct multiplication show that $(I - A)B = B(I - A) = I$.