Mathematics 700 Homework
Due Wednesday, October 23

Problem 1. Let $\sigma^1, \sigma^2, \dots, \sigma^n$ be a basis for $\mathbf{F}^{n*}$, which we view as a space of row vectors. Let

$$S := \begin{bmatrix} \sigma^1 \\ \sigma^2 \\ \vdots \\ \sigma^n \end{bmatrix}$$

be the matrix with rows $\sigma^1, \sigma^2, \dots, \sigma^n$. Then show that the columns $v_1, v_2, \dots, v_n$ of S^{-1} are the basis of $\mathbf{F}^n$ dual to $\sigma^1, \sigma^2, \dots, \sigma^n$.

Problem 2. Let $\mathcal{P}_n$ be the vector space of polynomials over $\mathbf{F}$ of degree $\leq n$. Let $z_0, z_1, \dots, z_n$ be distinct elemnts of $\mathbf{F}$ and define $f_0, f_1, \dots, f_n \in \mathcal{P}_n^*$ by

$$f_i(p(x)) = p(z_i).$$

Show that $f_0, f_1, \dots, f_n$ are a basis of $\mathcal{P}_n^*$ and find the basis of $\mathcal{P}_n$ dual to this basis. Hint: In different language we have already done this problem. If we can find polynomials $\ell_0, \ell_1, \dots, \ell_n$ so that

$$\ell_i(z_j) = \delta_{ij}$$

then you should be able to show that $\ell_0, \ell_1, \dots, \ell_n$ is the required basis of $\mathcal{P}_n$ and use them to show that $f_1.f_2, \dots, f_n$ are independent.

Problem 3. On $\mathcal{P}_2$ define linear functionals $\Lambda_1, \Lambda_2, \Lambda_3$ by

$$\Lambda_i(p) := \int_0^1 p(x)x^i \, dx \quad \text{for} \quad 1 \leq i \leq 3.$$

Show $\Lambda_1, \Lambda_2, \Lambda_3$ is a basis for $\mathcal{P}_2^*$ and find the basis of $\mathcal{P}_2$ dual to this basis.