Mathematics 700 Homework
Due Monday, October 21

The first several problems are to give you some basic facts about companion matrices in the special case $n = 4$. All of this generalizes to all $n \geq 1$, but let's start concretely. Let e_1, e_2, e_3, e_4 be the standard basis of $\mathbf{F}^4$. That is

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be a monic (that is lead coeffiecent one) polynomial. The ***companion matrix*** of $p(x)$ is the matrix

$$C := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}$$

Problem 1. Show $\operatorname{char}_C(x) = p(x)$. That is $p(x)$ is the characteristic polynomial of C. □

Let

$$v(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix} = e_1 + \lambda e_2 + \lambda^2 e_3 + \lambda^3 e_4.$$

Problem 2. Show

$$Cv(\lambda) = \lambda v(\lambda) - p(\lambda)e_4$$

and thus if $p(\lambda) = 0$, then $v(\lambda)$ is an eigenvector for C with eigenvalue λ. □

Problem 3. If $p(x)$ has four distinct root λ_1, λ_2, λ_3, and λ_4, and let P be the matrix with columns $v(\lambda_1), v(\lambda_2), v(\lambda_3), v(\lambda_4)$, that is,

$$P = [v(\lambda_1), v(\lambda_2), v(\lambda_3), v(\lambda_4)]$$

(this is our old friend the Vandermonde matrix) then

$$P^{-1}CP = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}.$$

□

Here are some problems related to linear functionals and hyperplanes. (The relation between the two being that hyperplanes are the kernels of nonzero linear functionals and the kernel of a linear functional determines it up to multiplication by a scalar.)

Problem 4. Find the basis of $\mathbf{R}^{3*}$ dual to the basis

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}.$$

of $\mathbf{R}^3$. □

Problem 5. Let W be a hyperplane in the finite dimensional vector space V and let $v_0 \in V$ with $v_0 \notin W$. Then show every vector $v \in V$ can be uniquely written as $v = w + cv_0$ where $w \in W$ and $c \in \mathbf{F}$ is a scalar. □

Problem 6. Let V be a finite dimensional vector and $f \in V^*$ a non-zero linear functional. Let $g\colon V \to \mathbf{F}$ be linear so that $\ker f \subseteq \ker g$. Then show there is a scalar $c \in \mathbf{F}$ with $g = cf$. Hint: Let $W := \ker f$. Then W is a hyperplane in V. Let $v_0 \in V$ with $v_0 \notin W$. Let $c = g(v_0)/f(v_0)$. Then use the last problem to show $g = cf$. □ □

Problem 7. Let $M_{2\times 2}(\mathbf{R})$ be the vector space of 2×2 matrices over the real numbers $\mathbf{R}$. Let $f\colon M_{2\times 2}(\mathbf{R}) \to \mathbf{R}$ be a linear functional so $f(AB) = f(BA)$ for all $A, B \in M_{2\times 2}(\mathbf{R})$. Show there is a scalar $c \in \mathbf{R}$ so that $f(A) = c\operatorname{tr}(A)$ for all $A \in M_{2\times 2}(\mathbf{R})$. Hint: Both f and tr are linear functionals on $M_{2\times 2}(\mathbf{R})$ so it is enough to show $\ker(\operatorname{tr}) \subseteq \ker(f)$. Show $\ker(\operatorname{tr}) = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. Now $f(AB) = f(BA)$ and linearity imply $f(AB - BA) = 0$ for all $A, B \in M_{2\times 2}(\mathbf{R})$. Use this to show $f(C) = 0$ for $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. (You should compare this with Problem 2 on the January 1985 qualifying exam.) □