Some more on matrices of linear maps. The following shows that it is possible to choose a basis for the range and domain of a linear map that puts makes the matrix particularly simple.

**Problem 1.** Let $V$ and $W$ be finite dimensional vector spaces and in let $T: V \to W$ be a linear.

(1) First assume that $\dim V = 5$, $\dim W = 6$ and that $\text{rank } T = 3$. Then by rank plus nullity we have $\text{nullity } T = 2$. Choose a basis $v_4, v_5$ of $\ker T$. Then these can be extended to a basis $V = \{ v_1, v_2, v_3, v_4, v_5 \}$ of $V$ (so the last two in the list are the basis of $\ker T$.) Let $w_1 = Tv_1, w_2 = Tv_2, w_3 = Tv_3$. Than as $\text{Span}\{ v_1, v_2, v_3 \} \cap \ker T = \{ 0 \}$ it follows that $w_1, w_2, w_3$ are linearly independent. Thus we can extend $\{ w_1, w_2, w_3 \}$ to a basis $W = \{ w_1, w_2, w_3, w_4, w_5, w_6 \}$ of $W$. Then show that in these bases the matrix of $T$ is

$$[T]_{V,W} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

(2) Now assume that $\dim V = m$, $\dim W = n$, and that $\text{rank } T = r$. Then show that it is possible to choose bases $V$ for $V$ and $W$ for $W$ so that the matrix $[T]_{V,W}$ has $r$ ones down the main diagonal and all other element zero.

**Quotient Spaces.** If there is any one idea the characterizes modern algebra it is the idea of a quotient structure. The following problems introduce the linear algebra version of this concept. You may have seen other versions as in the integers mod $n$, the quotient of a group by a normal subgroup, and the quotient of a ring by an ideal.

Let $V$ be a vector space over the field $F$ and $W$ a subspace of $V$. Then define an equivalence relation $\sim_W$ by

$$v_1 \sim_W v_2 \text{ if and only if } v_2 - v_1 \in W.$$ 

**Problem 1.** Show that this is an equivalence relation. (Recall a relation $\sim$ on a set $V$ is an equivalence relation iff the three conditions (1) $x \sim x$ for all $x \in V$ (it is reflexive) (2) $x \sim y$ implies $y \sim x$ for all $x, y \in V$ (it is symmetric) and (3) $x \sim y$ and $y \sim z$ implies $x \sim z$ (it is transitive) hold.)

Denote by $[v]_W$ the equivalence class of $v \in V$ under the equivalence relation $\sim_W$. That is

$$[v]_W := \{ u \in V : u \sim_W v \}.$$
Problem 2. Show $[v]_W = v + W$ where $v + W = \{v + w : w \in W\}$.

Let $V/W$ be the set of all equivalence classes of $\sim_W$. That is

$$V/W := \{[v]_W : v \in V\} = \{v + W : v \in V\}.$$  

The equivalence class $[v]_W = v + W$ is often called the coset of $v$ in $V/W$.

Problem 3. Let $V = \mathbb{R}^2$ and let $W$ be the subspace of points of $V$ of points $(x, y)$ with $y = -2x$. Then draw pictures of the coset of $(1, 1)$ in $V/W$ and the coset of $(3, -2)$ in $V/W$. What is a geometric description of the coset of $v \in \mathbb{R}^2$ in $V/W$?

Define a sum and scalar multiplication in $V/W$ by

$$[v_1]_W + [v_2]_W := [v_1 + v_2]_W, \quad c[v]_W := [cv]_W$$

where $v_1, v_2, v \in V$ and $c \in \mathbb{F}$.

Problem 4. Show this is well defined. Recall the term well defined is used in mathematics to mean “is independent of the choices made in the definition”. In this particular case this means you need to show

$$[v_1]_W = [v_1']_W \quad \text{and} \quad [v_2]_W = [v_2']_W \quad \text{implies} \quad [v_1 + v_2]_W = [v_1' + v_2']_W$$

and

$$[v]_W = [v']_W \quad \text{implies} \quad [cv]_W = [cv']_W.$$  

Proposition 1. With these operations $V/W$ is a vector space.

Problem 5. Prove this.

Define a map $\pi : V \to V/W$ by $\pi(v) = [v]_W$. (Or in slightly different notation $\pi v = v + W$.) This is the natural projection or canonical projection of $V$ onto the quotient space $V/W$.

Problem 6. The natural projection $\pi : V \to V/W$ is a linear map and $\ker(\pi) = W$.

Problem 7. If $V$ is finite dimensional then what is the dimension of $V/W$ in terms of $\dim V$ and $\dim W$? Prove your answer is correct. (HINT: Rank plus nullity.)

Problem 8. In the example of Problem 3 draw some pictures of cosets $v_1 + W$ and $v_2 + W$ what their sum $(v_1 + W) + (v_2 + W)$ and the linear combination $2(v_1 + W) − 3(v_2 + W)$ for a few choices of $v_1$ and $v_2$.

We now give a very basic result. In the context of groups this is called the first homomorphism theorem.

Theorem 1. Let $V$ and $U$ be vector spaces and let $T : V \to U$ be a surjective (that is onto) linear map. Then the vector space $V/\ker T$ is isomorphic to $U$. (Written as $U \approx V/\ker T$.)
We are not assuming that $V$ and $U$ are finite dimensional so it is not enough to just compute dimensions.

**Problem 9.** Prove this along the following lines: First to simplify notation set $W := \ker T$ so that

$$W = \{ v \in V : Tv = 0 \}.$$ 

Then we wish to show that $V/W \approx U$. Define a map $\tilde{T} : V/W \to U$ by

$$\tilde{T}[v]_W = Tv.$$

(a) Show that $\tilde{T}$ is well defined. (That is if $[v_1]_W = [v_2]_W$ then $Tv_1 = Tv_2$.)

(b) Show that $\tilde{T}$ is linear.

(c) Show that $\tilde{T}$ is onto (i.e. surjective). (**Hint:** Use that $T$ is onto so that every $u \in U$ is of the form $u = Tv$ for some $v \in V$.)

(d) Show that $\tilde{T}$ is one to one (i.e. injective). (**Hint:** You wish to show that if $\tilde{T}[v]_W = 0$ then $[v]_W = [0]_W$. But $[v]_W = [0]_W$ if and only if $v \in W = \ker T$.)