Mathematics 700, Test #2

Show your work to get credit. An answer with no work will not get credit.

1. Find the Smith normal form over the integers of the matrix

\[
A = \begin{bmatrix}
4 & 6 \\
8 & 10 \\
14 & 12
\end{bmatrix}.
\]

2. Find the invariant factors of the following matrices.

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \quad \text{(with } b \neq 0).\]

3. Let \( \mathcal{P}_1 = \text{Span}\{1, x\} \) be the real polynomials of degree \( \leq 1 \) with real coefficients and define two linear functionals \( \Lambda_1, \Lambda_2: \mathcal{P}_1 \to \mathbb{R} \) by

\[
\Lambda_1(p) := \int_0^1 p(x) \, dx, \quad \Lambda_2(p) = \int_0^1 xp(x) \, dx.
\]

Find the basis of \( \mathcal{P}_1 \) that is dual to \( \{\Lambda_1, \Lambda_2\} \).

4. Let \( A \) be an \( n \times n \) matrix with real entries so that \( A^t = A^{-1} \). Then show that \( \det(A) = \pm 1 \).

5. If \( T: V \to V \) is a linear operator on the vector space \( V \) that satisfies \( T^2 = I \), then show that the only eigenvalues of \( T \) are \( 1 \) and \( -1 \).

6. Let \( D \) be an invertible \( n \times n \) matrix and \( N \) a \( n \times n \) matrix so that \( DN = ND \) and \( N^3 = 0 \). Show that \( D + N \) is invertible.

7. Let \( A \) be a real \( 2 \times 2 \) matrix so that \( A^2 - 3A + 2I_2 = 0 \). Show that \( A \) is similar to one of the following three matrices

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}.
\]

8. Let \( A \) be an \( n \times n \) matrix over the reals with \( \det(A) \neq 0 \). Show that

\[
\det(\text{adj}(A)) = \det(A)^{n-1}.
\]

HINT: Recall that \( A \text{adj}(A) = \det(A)I \).
Instructions

This is to be done in three hours in one setting. I would prefer that it is closed book, but if you feel that you have to look up something write me a note like “I looked up the definition of rank and used it in problems numbers 31 and 57”. I will then take a little off on these problems, say 20%. It would also make my life a little easier if you worked each problem on a separate sheet of paper. This is due in class Monday November 21.

As to studying for it, here a couple of our recent topics you should certainly know before taking the exam. How to find the Smith normal form of a matrix over a Euclidean domain. How to find the invariant factors of a matrix over a field.