Mathematics 700, Test #2

Show your work to get credit. An answer with no work will not get credit.

1. Find the Smith normal form over the integers of the matrix
$$A = \begin{bmatrix} 4 & 6 \\ 8 & 10 \\ 14 & 12 \end{bmatrix}.$$
2. Find the invariant factors of the following matrices.
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ (with } b \neq 0).$$
3. Let $\mathcal{P}_1 = \operatorname{Span}\{1, x\}$ be the real polynomials of degree ≤ 1 with real coefficients and define two linear functionals $\Lambda_1, \Lambda_2 \colon \mathcal{P}_1 \to \mathbf{R}$ by
$$\Lambda_1(p) := \int_0^1 p(x)\,dx, \quad \Lambda_2(p) = \int_0^1 xp(x)\,dx.$$
Find the basis of $\mathcal{P}_1$ that is dual to $\{\Lambda_1, \Lambda_2\}$.
4. Let A be an $n \times n$ matrix with real entries so that $A^t = A^{-1}$. Then show that $\det(A) = \pm 1$.
5. If $T \colon V \to V$ is a linear operator on the vector space V that satisfies $T^2 = I$, then show that the only eigenvalues of T are 1 and -1.
6. Let D be an invertible $n \times n$ matrix and N a $n \times n$ matrix so that $DN = ND$ and $N^3 = 0$. Show that $D + N$ is invertible.
7. Let A be a real 2×2 matrix so that $A^2 - 3A + 2I_2 = 0$. Show that A is similar to one of the following three matrices
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$
8. Let A be an $n \times n$ matrix over the reals with $\det(A) \neq 0$. Show that
$$\det(\operatorname{adj}(A)) = \det(A)^{n-1}.$$
Hint: Recall that $A \operatorname{adj}(A) = \det(A) I$.

Instructions

This is to be done in three hours in one setting. I would prefer that it is closed book, but if you feel that you have to look up a something write me a note like "I looked up the definition of rank and used it in problems numbers 31 and 57". I will then take a little off on these problems, say 20%. It would also make my life a little easier if you worked each problem on a separate sheet of paper. This is due in class Monday November 21.

As to studying for it, here a couple of our recent topics you should certainly know before taking the exam. How to find the Smith normal form of a matrix over a Euclidean domain. How to find the invariant factors of a matrix over a field.