Mathematics 700 Test #1 Name:______________________________

Show your work to get credit. An answer with no work will not get credit.

1. (15 Points) Define the following:
   (a) Linear independence.

   (b) The span of a subset S of a vector space V.

   (c) The vector space V is direct sum of its subspaces U and W.

2. (10 Points) Find (no proof required) a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ so that

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \qquad T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

3. (10 Points) Let v_1, v_2, v_3 be linearly independent vectors in a vector space V. Then show that the vectors $v_1, 2v_1 + v_2, 3v_1 + 2v_2 + v_3$ are also linearly independent.

4. (10 Points) Let $M_{2\times 2}$ be the 2 by 2 matrics over the field $\mathbb{F}$ and let

$$\mathcal{D} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{F} \right\}$$

be the subspace of diagonal matrices. Show that any three dimensional subspace of $M_{2\times 2}$ contains a nonzero diagonal matrix.

5. (10 Points) Find (no proof required) a basis for the set of the space of vectors $(x, y, z, w) \in \mathbb{R}^4$ that satisfy

$$\begin{aligned} x + y + z + w &= 0 \\ x + y + 2z + 3w &= 0. \end{aligned}$$

6. (15 Points) Show that if $v_1, \ldots, v_k$ are vectors in the vector space V and $c_1, \ldots, c_k \in \mathbb{F}$ are scalars so that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0, \quad \text{and} \quad c_k \neq 0$$

then

$$\operatorname{Span}\{v_1, \ldots, v_{k-1}\} = \operatorname{Span}\{v_1, \ldots, v_k\}.$$

7. (15 Points) Let U and W be subspaces of a vector space so that

$$\dim U = 3, \quad \dim W = 4, \quad \dim(U \cap W) = 2.$$

Then show directly, that is without using the theorem that $\dim(U+W) = \dim U + \dim W - \dim U \cap W$, that $\dim(U+W) = 5$. (So you are being ask to prove $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$ in this special case.)

8. (15 Points) Let $\mathcal{U} = \{u_1, u_2\}$ and $\mathcal{W} = \{w_1, w_2, w_3\}$ be two linearly independent sets in a vector space V such that $\mathcal{U} \cup \mathcal{W}$ is linearly independent. Then show

$$\text{Span}(\mathcal{U}) \cap \text{Span}(\mathcal{W}) = \{0\}.$$