1. Some Bases of the Polynomials of Degree \( \leq n \).

Let \( \mathcal{P}_n \) be the vector space of real polynomials of degree \( \leq n \). That is
\[
\mathcal{P}_n := \{ a_0 + a_1 x + \cdots + a_n x^n : a_0, \ldots, a_n \in \mathbb{R} \}.
\]
We have seen that \( \dim \mathcal{P}_n = n + 1 \) and that it has the “usual” basis
\[
U := \{1, x, \ldots, x^n\}
\]
coming from the powers of \( x \). In different problems there different bases of \( \mathcal{P}_n \) that are better adapted to the problem. For example let \( a \in \mathbb{R} \) then the ordered basis
\[
\mathcal{A} := \left\{1, (x - a), \frac{(x - a)^2}{2}, \frac{(x - a)^3}{3!}, \ldots, \frac{(x - a)^n}{n!}\right\}
\]
has the property that if \( f(x) \in \mathcal{P}_n \) is expressed in this basis:
\[
f(x) = \sum_{k=0}^{n} a_k \frac{(x - a)^k}{k!}
\]
then the coordinate \( a_0, \ldots, a_n \) are given by
\[
a_k = f^{(k)}(a)
\]
where \( f^{(k)} \) is the \( k \)-th derivative of \( f(x) \) (and \( f^{(0)} = f \) be definition).

**Problem 1.** Derive this formula for \( a_k \) by taking the \( k \)-th derivative of both sides of (1.1) and then setting \( x = a \). (Note this is exactly the usual derivation of the coefficients in a Taylor series that you know and love from calculus.)
In our terminology the coordinate vector of the vector \( f(x) \in \mathcal{P}_n \) is

\[
[f(x)]_A = \begin{bmatrix} f(a) \\ f'(a) \\ f''(a) \\ \vdots \\ f^{(n-1)}(a) \\ f^{(n)}(a) \end{bmatrix}
\]

which is just the list of the values of the derivatives of \( f(x) \) at \( x = a \). So in a context where one is working with the derivatives of polynomials at the point \( x = a \) the basis \( A \) is the natural one to use.

2. Formulas for Sums of Powers.

We now give a basis of \( \mathcal{P}_n \) were it is easy to derive “summation formulas” (the precise meaning of this will be cleared up below) and then by expressing the usual basis \( \{1, x, x^2, \ldots, x^n\} \) in terms of this basis we can derive formulas for sums of powers. This is a theme we will see repeatedly during the term: Make a problem easier by changing to a nicer basis. Set

\[ S_0(x) := 1 \]

and for \( 1 \leq k \leq n \) set

\[ S_k(x) := x(x-1)(x-2)\cdots(x-k+1). \]

This has \( k \) factors and so has degree \( k \). For small values of \( k \) we have

\[ S_0(x) = 1, \]
\[ S_1(x) = x, \]
\[ S_2(x) = x(x-1), \]
\[ S_3(x) = x(x-1)(x-2), \]
\[ S_4(x) = x(x-1)(x-2)(x-3). \]

Let

\[ \mathcal{S} := \{S_0(x), S_1(x), \ldots, S_n(x)\}. \]

Then \( \mathcal{S} \) is an ordered basis of \( \mathcal{P}_n \). We now define the change of basis matrices between the usual basis \( \mathcal{U} \) and the basis \( \mathcal{S} \) by

\[ S_k(x) = \sum_{i=0}^{k} a_{ki} x^i \]
and

\[(2.1)\quad x^k = \sum_{i=0}^{k} b_{ki} S_i(x).\]

So for example
\[S_3(x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x = a_{33}x^3 + a_{32}x^2 + a_{31}x + a_{30}\]

which implies
\[a_{33} = 1, \quad a_{32} = -3, \quad a_{31} = 2, \quad a_{30} = 0.\]

Likewise
\[x^3 = S_3(x) + 3S_2(x) + S_1(x) = b_{33}S_3(x) + b_{32}S_2(x) + b_{31}S_1(x) + b_{30}S_0(x)\]
yielding
\[b_{33} = 1, \quad b_{32} = 3, \quad b_{31} = 1, \quad b_{30} = 0.\]

**Problem 2.** Show that \(AB = BA = I_{n+1}\) on \(P_n\). \(\Box\)

On \(P_6\) the matrices \(A = [a_{ki}]\) and \(B = [b_{ki}]\) are 7 by 7 matrices given explicitly by

\[
A = [a_{ki}] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 & 0 & 0 \\
0 & -6 & 11 & -6 & 1 & 0 & 0 \\
0 & 24 & -50 & 35 & -10 & 1 & 0 \\
0 & -120 & 274 & -225 & 85 & -15 & 1
\end{bmatrix}
\]

and

\[
B = [b_{ki}] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 3 & 1 & 0 & 0 & 0 \\
0 & 1 & 7 & 6 & 1 & 0 & 0 \\
0 & 1 & 15 & 25 & 10 & 1 & 0 \\
0 & 1 & 31 & 90 & 65 & 15 & 1
\end{bmatrix}
\]

In matrix notation this means
\[
\begin{bmatrix}
S_0(x) \\
S_1(x) \\
S_2(x) \\
S_3(x) \\
S_4(x) \\
S_5(x) \\
S_6(x)
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 & 0 & 0 \\
0 & -6 & 11 & -6 & 1 & 0 & 0 \\
0 & 24 & -50 & 35 & -10 & 1 & 0 \\
0 & -120 & 274 & -225 & 85 & -15 & 1
\end{bmatrix} \begin{bmatrix}
x \\
x^2 \\
x^3 \\
x^4 \\
x^5 \\
x^6
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
1 \\
x \\
x^2 \\
x^3 \\
x^4 \\
x^5 \\
x^6
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 3 & 1 & 0 & 0 & 0 \\
0 & 1 & 7 & 6 & 1 & 0 & 0 \\
0 & 1 & 15 & 25 & 10 & 1 & 0 \\
0 & 1 & 31 & 90 & 65 & 15 & 1
\end{bmatrix}
\begin{bmatrix}
S_0(x) \\
S_1(x) \\
S_2(x) \\
S_3(x) \\
S_4(x) \\
S_5(x) \\
S_6(x)
\end{bmatrix}.
\]

**Problem 3.** Check that the the values for \(a_{ki}\) and \(b_{ki}\) are correct for \(i, k \leq 4\). \(\square\)

There are some obvious patterns in these matrices.

**Problem 4.** Show the following:

1. \(a_{kk} = b_{kk} = 1\) for all \(k \geq 0\).
2. \(a_{k0} = b_{k0} = 0\) for \(k \geq 1\).
3. The signs in the matrix \(A\) have a chess board pattern. That is \((-1)^{i+k}a_{ki} \geq 0\).
4. \(a_{k,k-1} = -b_{k,k-1}\). (Not easy, use that \(AB = I\).)
5. \(b_{ki} \geq 0\) for all \(i, k\). (Hard.)
6. \(b_{k1} = 1\) for \(k \geq 1\). (Hard.) \(\square\)

From the point of view of summation formulas what makes the \(S_k(x)\) nice is the relation:

\[
S_k(x) = \frac{1}{k+1}(S_{k+1}(x + 1) - S_{k+1}(x))
\]

which holds for \(k \geq 0\).

**Problem 5.** Verify this formula. \(\square\)

The standard trick with telescoping series shows that

\[
S_k(0) + S_k(1) + S_k(2) + \cdots + S_k(N) = \frac{1}{k+1}(S_{k+1}(N + 1) - S_{k+1}(0))
= \frac{1}{k+1}S_{k+1}(N + 1).
\]

(At the last step we have used \(S_{k+1}(0) = 0\).)

**Problem 6.** Verify this. \(\square\)

In summation notation this is

\[
(2.2) \quad \sum_{j=0}^{N} S_k(j) = \frac{1}{k+1}S_{k+1}(N + 1).
\]
Now we can give formulas for sums of powers of integers. Using equations (2.1) and (2.2) we have
\[
\sum_{j=0}^{N} j^k = \sum_{j=0}^{N} \sum_{i=0}^{k} b_{ki} S_i(j) \\
= \sum_{i=0}^{k} b_{ki} \sum_{j=0}^{N} S_i(j) \\
= \sum_{i=0}^{k} b_{ki} \frac{1}{i+1} S_{i+1}(N+1).
\]

For small values of \(k\) this gives
\[
\sum_{j=0}^{N} j = b_{11} \frac{1}{2} S_2(N+1) + b_{10} S_1(N+1) \\
\sum_{j=0}^{N} j^2 = b_{22} \frac{1}{3} S_3(N+1) + b_{21} \frac{1}{2} S_1(N+1) + b_{20} S_0(N+1) \\
\sum_{j=0}^{N} j^3 = b_{33} \frac{1}{4} S_3(N+1) + b_{32} \frac{1}{3} S_1(N+1) + b_{31} \frac{1}{2} S_0(N+1) + b_{30} S_0(N+1).
\]

**Problem 7.** Use these to derive the familiar formulas for \(\sum_{j=0}^{N} j\), \(\sum_{j=0}^{N} j^2\), and \(\sum_{j=0}^{N} j^3\). \(\square\)