1. Define the following:
   (a) eigenvalue.
   Let $T: V \to V$ be linear map (where $V$ is a vector space over the field of scalars $\mathbb{F}$). Then $\lambda \in \mathbb{F}$ is an eigenvalue iff there is nonzero vector $v \in V$ so that $Tv = \lambda v$.
   REMARK: It was popular to define an eigenvalue to be a root of the characteristic equation $\det(\lambda I - T) = 0$. That such roots are the eigenvalues is the conclusion of a theorem, not the definition.
   (b) eigenvector.
   Let $T: V \to V$ be linear map (where $V$ is a vector space over the field of scalars $\mathbb{F}$). Then $v \in V$ is an eigenvector iff $v$ is nonzero and there is a scalar $\lambda \in \mathbb{F}$ so that $Tv = \lambda v$.
   (c) coordinates of a vector relative to a basis.
   Let $V$ have $\mathcal{V} = \{v_1, \ldots, v_n\}$ as an ordered basis. Then any vector $x \in V$ can be uniquely written as $x = x_1v_1 + x_2v_2 + \cdots x_nv_n$ were $x_1, x_2, \ldots, x_n \in \mathbb{F}$. Then the column vector
   \[
   [x]_\mathcal{V} = \begin{bmatrix}
   x_1 \\
   x_2 \\
   \vdots \\
   x_n
   \end{bmatrix}
   \]
   is the coordinate vector of $x$ in the basis $\mathcal{V}$ and the scalars $x_1, x_2, \ldots, x_n$ are the coordinates of $x$ in this basis.
   (d) the three elementary row operations.
   The elementary row operations over a commutative ring $R$ are:
   (i) Multiplying a row by an invertible element of $R$.
   (ii) Interchanging two rows.
   (iii) Adding a scalar multiple of one row to another row.
   (e) the Smith Normal Form of a matrix over a Euclidean domain.
   Let $A \in M_{m \times n}(R)$. Then $F$ is a the Smith normal form of $A$ iff $F$ is equivalent to $F$ by elementary row and column operations and $F$ is diagonal of the form
   \[
   F = \begin{bmatrix}
   f_1 \\
   f_2 \\
   \cdots \\
   f_r \\
   0 \\
   \cdots
   \end{bmatrix}
   \]
   where the elements $f_1, f_2, \ldots, f_r$ are nonzero and satisfy $f_1 \mid f_2 \mid \cdots \mid f_{r-1} \mid f_r$.
   (f) $V$ is the direct sum of $W_1, \ldots, W_k$.
   The vector space $V$ is a direct sum of $W_1, \ldots, W_k$ iff $W_1 \cup \cdots \cup W_k$ spans $V$ and every element $v \in V$ can be uniquely expressed as $v = w_1 + w_2 + \cdots + w_k$ with $w_i \in W_i$.

2. Let $V$ be a vector space and let $v_1, \ldots, v_n \in V$ be linearly independent vectors. Let $v \in V$ be a vector so that $v \notin \text{Span}\{v_1, \ldots, v_n\}$. Show that $\{v_1, \ldots, v_n, v\}$ is a linearly independent set.
Solution: To show that \(\{v_1, \ldots, v_n, v\}\) is a linearly independent set we need to show that if the linear combination

\[
a_1v_1 + \cdots + a_nv_n + av = 0
\]

then all of the scalars \(a_1, \ldots, a_n, a\) are zero. First we note that \(a = 0\), for if \(v\) not we can solve \(a_1v_1 + \cdots + a_nv_n + av = 0\) for \(v\) to get

\[v = \frac{-a_1}{a}v_1 + \frac{-a_2}{a}v_2 + \cdots + \frac{-a_n}{a}v_n \in \text{Span}\{v_1, \ldots, v_n\},\]

contradicting the assumption that \(v \notin \text{Span}\{v_1, \ldots, v_n\}\). Using \(a = 0\) in (1) implies

\[a_1v_1 + \cdots + a_nv_n = 0.
\]

But as \(\{v_1, \ldots, v_n\}\) is linearly independent this implies that \(a_1 = a_2 = \cdots a_n = 0\) as required. \(\square\)

3. Let \(M_{k \times k}(\mathbb{R})\) be the vector space of \(k \times k\) matrices over the real numbers. \(T: M_{3 \times 3}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})\) be linear. Show there is a nonzero symmetric matrix, \(A\), so that \(T(A) = 0\). (Recall that \(A\) is symmetric iff \(A^t = A\) where \(A^t\) is the transpose of \(A\).)

Solution: Let \(S \subset M_{3 \times 3}(\mathbb{R})\) be the subspace of \(M_{3 \times 3}(\mathbb{R})\) of symmetric matrices. Then \(\dim S = 6\) as it as for a basis

\[
\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.
\]

Let \(S = T|_S\) be the restriction of \(T\) to \(S\). Then \(S: S \to M_{2 \times 2}(\mathbb{R})\) and \(\dim M_{2 \times 2}(\mathbb{R}) = 4\). Therefore \(\text{rank } S + \text{nullity } S = 6\) implies that \(\text{nullity } S = \dim \ker S \geq 2\). Therefore \(\ker S \neq \{0\}\). So letting \(A\) be a nonzero element of \(\ker S \subseteq S\) and using that \(S\) is the restriction of \(T\) we see that \(T(A) = S(A) = 0\). Thus \(A\) is a nonzero symmetric matrix with \(T(A) = 0\). \(\square\)

4. What is the Jordan canonical form, over the complex numbers, of a matrix that has elementary divisors \(x - 1, (x - 1)^2, x^2 - 6x + 25, (x^2 - 6x + 25)^2\)?

Solution: Over the complex numbers we have the factorization \(x^2 - 6x + 25 = (x - (3 + 4i))(x - (3 - 4i))\). Therefore the Jordan canonical form will consist of the block diagonal matrix with the Jordan blocks for \((x - 1), (x - 1)^2, (x - (3 + 4i)), (x - (3 + 4i))^2, (x - (3 - 4i)),\) and \((x - (3 - 4i))^2\) along the diagonal. That is the Jordan form is

\[
\begin{bmatrix}
1 & 1 & 1 \\
& 3 + 4i & \\
& 3 + 4i & 1 \\
& 3 + 4i & 3 - 4i \\
& & 3 - 4i \\
& & 3 - 4i
\end{bmatrix}
\]

where all unspecified elements are zero. \(\square\)
5. Let 

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 \\ 3 & 4 & 1 & 1 \end{bmatrix} \]

Then for \( A \) find 
(a) The invariant factors. 

**Solution:** By definition the invariant factors of \( A \) are the invariant factors of \( xI - A \) over the Euclidean domain \( \mathbb{R}[x] \). As this is block lower triangular we have 

\[
\det(xI - A) = \det \begin{bmatrix} x & -1 \\ -1 & x \end{bmatrix} \det \begin{bmatrix} x - 1 & 1 \\ -1 & x - 1 \end{bmatrix} \\
= (x^2 - 1)((x - 1)^2 + 1) \\
= (x - 1)(x + 1)(x^2 + 2x + 2).
\]

This has distinct prime factors so the elementary divisors of \( xI - A \) are \( f_1 = f_2 = f_3 = 1 \) and \( f_4 = (x - 1)(x + 1)(x^2 + 2x + 2) \). (This is because \( f_1 | f_2 | f_3 | f_4 \) and \( f_1 f_2 f_3 f_4 = (x - 1)(x + 1)(x^2 + 2x + 2) \)). So if \( f_3 \neq 1 \), then the then \( f_3^2 | (x - 1)(x + 1)(x^2 + 2x + 2) \) so that the square of some non-constant polynomial would divide \( (x - 1)(x + 1)(x^2 + 2x + 2) \). As this is not the case we have \( f_3 = 1 \). \( \square \)

(b) The elementary divisors. 

**Solution:** The elementary divisors are the prime power factors of the invariant factors. So in this case they are \( q_1 = x - 1 \), \( q_2 = x + 1 \), \( q_3 = x^2 - 2x + 2 \). \( \square \)

(c) The minimal polynomial. 

**Solution:** The minimal polynomial is just the invariant factor of largest degree. That is \( \min_A(x) = f_x = (x - 1)(x + 1)(x^2 + 2x + 2) \). \( \square \)

(d) The rational canonical form over the reals. 

**Solution:** The rational canonical form is the block diagonal matrix with the companion matrices of the elementary divisors along the diagonal. This is 

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\]

6. Let \( V \) be a finite dimensional vector space over the real numbers \( \mathbb{R} \) and let \( W \) be a subspace of \( V \). Assume that there are vectors \( v_1, v_2 \in V \) so that 

\[ v_1 \notin W \quad \text{and} \quad v_2 \notin \text{Span}(\{v_1\} \cup W). \]

There show there is a linear functional \( f \in V^* \) so that \( f(w) = 0 \) for all \( w \in W \), \( f(v_1) = 1 \) and \( f(v_2) = 2 \).

**Solution:** We know that a given in finite dimensional vector spaces we can separate a vector not in a subspace form a subspace by a linear functional. This means that we can find linear functionals \( f_1, f_2 \in V^* \) so that 

\[ f_1(v_1) = 1, \quad \text{and} \quad f_1(w) = 0 \quad \text{for all} \quad w \in W \]

and 

\[ f_2(v_2) = 1 \quad \text{and} \quad f_2(x) = 0 \quad \text{for all} \quad x \in \text{Span}(\{v_1\} \cup W). \]
Let $f$ be the linear combination of $f_1$ and $f_2$ given by
\[ f = f_1 + (2 - f_1(v_2))f_2. \]
(Note that $(2 - f_1(v_2))$ is a scalar so this is a linear combination.) Then if $w \in W$ we have
\[ f(w) = f_1(w) + (2 - f_1(v_2))f_2(w) = 0 \]
as $f_1$ and $f_2$ are both zero on $W$. Next
\[ f(v_1) = f_1(v_1) + (2 - f_1(v_2))f_2(v_1) = 1 + 0 = 1 \]
as $f_1(v_1) = 1$ and $f_2(v_1) = 0$. And to finish
\[ f(v_2) = f_1(v_1) + (2 - f_1(v_2))f_2(v_2) = f_1(v_1) + (2 - f_1(v_2)) = 2 \]
as $f_2(v_2) = 1$. □

7. Let $V$ be a finite dimensional vector space over the complex numbers and $P : V \to V$ a linear map so that $P^2 = P$. Show that trace $P = \text{rank } P$

**Solution:** We know that if $P^2 = P$ (that is $P$ is a projection) that $V = \text{image } P + \ker P$. (This can be seen directly as for $v \in V$ we have $v = P(v) + (v - P(v))$ and $P(v - P(v)) = P(v - P^2 v) = P(v) - Pv = 0$ so that $v - P(v) \in \ker P$. Therefore image $P + \ker P = V$. But if $v \in \text{image } P \cap \ker P$ then we have $v = Px$ for some $x$. Then $Pv = P^2x = Px = v$. Therefore, as also $v \in \text{image } P$, $v = P(v) = 0$. Thus image $P \cap \ker P = \{0\}$. But image $P + \ker P = V$ and image $P \cap \ker P = \{0\}$ imply that $V = V = \text{image } P + \ker P$. Let $n = \dim V$ and $r = \dim \text{image } P = \text{rank } P$ and choose a basis $v_1, \ldots, v_r$ of image $P$ and then a basis of $v_{r+1}, \ldots, v_n$ of $\ker P$. Then $\{v_1, \ldots, v_n\}$ is a basis of $V$. If $1 \leq i \leq r$ then $v_i \in \text{image } P$ so that $v_i = Px_i$ for some $x_i \in V$. But then $Pv_i = P^2x_i = Px_i = v_i$. If $r + 1 \leq j \leq n$ then $v_j \in \ker P$ and so $Pv_j = 0$. Summarizing:

\[ P v_i = \begin{cases} v_i, & 1 \leq i \leq r; \\ 0, & r + 1 \leq i \leq n. \end{cases} \]

This implies the matrix of $P$ in the basis $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the block matrix
\[ [P]_{\mathcal{V}} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \]

where $I_r$ is the $r \times r$ identity matrix. But then trace $P = \text{trace} [P]_{\mathcal{V}} = r = \text{rank } P$. □

8. Let $\mathcal{P}_2 = \text{Span}\{1, x, x^2\}$ be the real polynomials of degree $\leq 2$. Define $T : \mathcal{P}_2 \to \mathcal{P}_2$ by
\[ T(p)(x) = p(x + 1). \]

Let $\mathcal{P}_2^*$ be the dual space to $\mathcal{P}_2$ and let $\Lambda \in \mathcal{P}_2^*$ be the functional
\[ \Lambda(p) = p(-3). \]

Then compute $\langle x^2, T^* \Lambda \rangle$.

**Solution:** This is just a case through the definitions:
\[ \langle x^2, T^* \Lambda \rangle = \langle Tx^2, \Lambda \rangle = \langle (x + 1)^2, \Lambda \rangle = (-3 + 1)^2 = 4. \]
9. A $5 \times 5$ real matrix $A$ has minimal polynomial $\min_A(x) = (x-1)^2 (x-2) (x-3)$ and trace($A$) = 10. What is the rational canonical form for $A$?

**Solution:** Let $f_1, \ldots, f_5$ be the invariant factors of $A$. Then $f_5$ is the minimal polynomial of $A$ and so $f_5 = (x-1)^2 (x-2) (x-3)$. We also know that the product $f_1 f_2 f_3 f_4 f_5 = \text{char}_A(x) = \det(xI - A)$ which has degree 5. Therefore the only possibilities for the other invariant factors are $f_1 = f_2 = f_3 = 1$ and $f_5$ is one of $x-1$, $x-2$, $x_3$. Let $f_4 = x - a$ were $a$ (which is either 1, 2, or 3) is to be determined. Then the elementary divisors of $A$ will be $(x-1)^2 = x^2 - 2x + 1$, $x-2$, $x-3$ and $x-a$ so that the rational canonical form is

$$
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & a
\end{bmatrix}
$$

The trace of this is $0 + 2 + 2 + 3 + a = 7 + a$ and we are given that the trace is 10. Therefore $a = 3$ and the rational canonical form of $A$ is

$$
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix}
$$

10. Let $V$ and $W$ be vector spaces over the field $\mathbf{F}$ and let $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$ so the following two conditions hold

(a) Span$\{v_1, \ldots, v_n\} = V$ (but we do not assume $\{v_1, \ldots, v_n\}$ is linearly independent),

(b) For any scalars $c_1, \ldots, c_n \in \mathbf{F}$

$$
\sum_{k=1}^{n} c_k v_k = 0 \quad \text{implies} \quad \sum_{k=1}^{n} c_k w_k = 0.
$$

Show that there is a unique linear map $T : V \to W$ so that $Tv_k = w_k$ for $1 \leq k \leq n$.

**Solution:** By reordering we can assume that $v_1, \ldots, v_k$ is a basis of $V$ where $k \leq n$. Then for $k+1 \leq j \leq n$ we have that $v_j \in \text{Span}\{v_1, \ldots, v_k\}$. By our basic existence theorem for linear maps we that that there is a unique linear map $T : V \to W$ so that

$$
Tv_i = w_i \quad \text{for} \quad 1 \leq i \leq k.
$$

(Note that if the required map exists that it must satisfy this condition. This shows that if $T$ exists, then it is unique.) For $k+1 \leq j \leq n$ we have that $v_j \in \text{Span}\{v_1, \ldots, v_k\}$ there are scalars $c_1, \ldots, c_k$ so that $v_j = c_1 v_1 + \cdots + c_k v_k$. Therefore

$$
Tv_j = T(c_1 v_1 + \cdots + c_k v_k) = c_1 Tv_1 + \cdots + c_k Tv_k = c_1 w_1 + \cdots + c_k w_k.
$$

But we also have

$$
v_j - c_1 v_1 - \cdots - c_k v_k = 0.
$$

By the condition (2) this implies $w_j - c_1 w_1 - \cdots - c_k w_k = 0$ so that

$$
w_j = c_1 w_1 + \cdots + c_k w_k.
$$
Now combining (3) and (4) gives that $Tv_j = w_j$. This shows that $Tv_i = w_i$ for $1 \leq i \leq n$ and completes the proof.