Mathematics 700 Final Name:________________________________

Show your work to get credit. An answer with no work will not get credit.

1. Define the following:
   (a) eigenvalue.
   Let $T\colon V \to V$ be linear map (where V is a vector space over the field of scalars $\mathbf{F}$). Then $\lambda \in \mathbf{F}$ is an ***eigenvalue*** iff there is nonzero vector $v \in V$ so that $Tv = \lambda v$. REMARK: It was popular to define an eigenvalue to be a root of the characteristic equation $\det(\lambda I - T) = 0$. That such roots are the eigenvalues is the conclusion of a theorem, not the definition.
   (b) eigenvector.
   Let $T\colon V \to V$ be linear map (where V is a vector space over the field of scalars $\mathbf{F}$). Then $v \in V$ is an ***eigenvector*** iff v is nonzero and there is a scalar $\lambda \in \mathbf{F}$ so that $Tv = \lambda v$.
   (c) coordinates of a vector relative to a basis.
   Let V have $\mathcal{V} = \{v_1, \ldots, v_n\}$ as an ordered basis. Then any vector $x \in V$ can be uniquely written as $x = x_1v_1 + x_2v_2 + \cdots x_nv_n$ were $x_1, x_2, \ldots, x_n \in \mathbf{F}$. Then the column vector

$$[x]_{\mathcal{V}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

   is the ***coordinate vector of*** x ***in the basis*** $\mathcal{V}$ and the scalars $x_1, x_2, \ldots, x_n$ are the ***coordinates of*** x in this basis.
   (d) the three elementary row operations.
   The elementary row operations over a commutative ring R are:
   (i) Multiplying a row by an invertible element of R.
   (ii) Interchanging two rows.
   (iii) Adding a scalar multiple of one row to another row.
   (e) the Smith Normal Form of a matrix over a Euclidean domain.
   Let $A \in M_{m\times n}(R)$. Then F is a the Smith normal form of A iff F is equivalent to F by elementary row and column operations and F is diagonal of the form

$$F = \begin{bmatrix} f_1 & & & & & \\ & f_2 & & & & \\ & & \ddots & & & \\ & & & f_r & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix}$$

   where the elemnts $f_1, f_2, \ldots, f_r$ are nonzero and satisfy $f_1 \mid f_2 \mid \cdots \mid f_{r-1} \mid f_r$.
   (f) V is the direct sum of $W_1, \ldots, W_k$.
   The vector space V is a ***direct sum*** of $W_1, \ldots, W_k$ iff $W_1 \cup \cdots \cup W_k$ spans V and every elment $v \in V$ can be uniquely expressed as $v = w_1 + w_2 + \cdots + w_k$ with $w_i \in W_i$.
2. Let V be a vector space and let $v_1, \ldots, v_n \in V$ be linearly independent vectors. Let $v \in V$ be a vector so that $v \notin \operatorname{Span}\{v_1, \ldots, v_n\}$. Show that $\{v_1, \ldots, v_n, v\}$ is a linearly independent set.

Solution: To show that $\{v_1, \dots, v_n, v\}$ is a linearly independent set we need to show that if the linear combination

$$a_1 v_1 + \cdots + a_n v_n + av = 0 \tag{1}$$

then all of the scalars $a_1, \dots, a_n, a$ are zero. First we note that $a = 0$, for iff not we can solve $a_1 v_1 + \cdots + a_n v_n + av = 0$ for v to get

$$v = \frac{-a_1}{a} v_1 + \frac{-a_2}{a} v_n + \cdots + \frac{-a_n}{a} v_n \in \operatorname{Span}\{v_1, \dots, v_n\},$$

contradiction the assumption that $v \notin \operatorname{Span}\{v_1, \dots, v_n\}$. Using $a = 0$ in (1) implies

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

But as $\{v_1, \dots, v_n\}$ is linearly independent this implies that $a_1 = a_2 = \cdots a_n = 0$ as required. □

3. Let $M_{k\times k}(\mathbf{R})$ be the vector space of $k \times k$ matrices over the real numbers. $T\colon M_{3\times 3}(\mathbf{R}) \to M_{2\times 2}(\mathbf{R})$ be linear. Show there is a nonzero symmetric matrix, A, so that $T(A) = 0$. (Recall that A is symmetric iff $A^t = A$ where A^t is the transpose of A.)

Solution: Let $\mathcal{S} \subset M_{3\times 3}(\mathbf{R})$ be the subspace of $M_{3\times 3}(\mathbf{R})$ of symmetric matrices. Then $\dim \mathcal{S} = 6$ as it as for a basis

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

Let $S = T\big|_{\mathcal{S}}$ be the restriction of T to $\mathcal{S}$. Then $S\colon \mathcal{S} \to M_{2\times 2}(\mathbf{R})$ and $\dim M_{2\times 2}(\mathbf{R}) = 4$. Therefore $\operatorname{rank} S + \operatorname{nullity} S = 6$ implies that $\operatorname{nullity} S = \dim\ker S \geq 2$. Therefore $\ker S \neq \{0\}$. So letting A be a nonzero element of $\ker S \subseteq \mathcal{S}$ and using that S is the restriction of T we see that $T(A) = S(A) = 0$. Thus A is a nonzero symmetric matrix with $T(A) = 0$. □

4. What is the Jordan canonical form, over the complex numbers, of a matrix that has elementary divisors $x - 1$, $(x-1)^2$, $x^2 - 6x + 25$, $(x^2 - 6x + 25)^2$?

Solution: Over the complex numbers we have the factorization $x^2 - 6x + 25 = (x - (3 + 4i))(x - (3 - 4i))$. Therefore the Jordan canonical form will consist of the block diagonal matrix with the Jordan blocks for $(x-1)$, $(x-1)^2$, $(x-(3+4i))$, $(x-(3+4i))^2$, $(x-(3-4i))$, and $(x-(3-4i))^2$ along the diagonal. That is the Jordan form is

$$\begin{bmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ & 1 & 1 &&&&&& \\ &&& 3+4i &&&&& \\ &&&& 3+4i &&&& \\ &&&& 1 & 3+4i &&& \\ &&&&&& 3-4i && \\ &&&&&&& 3-4i & \\ &&&&&&& 1 & 3-4i \end{bmatrix}$$

where all unspecified elements are zero. □

5. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 \\ 3 & 4 & 1 & 1 \end{bmatrix}$$

Then for A find

(a) The invariant factors.

Solution: By definition the invariant factors of A are the invariant factors of $xI - A$ over the Euclidean domain $\mathbf{R}[x]$. As this is block lower triangular we have

$$\begin{aligned} \det(xI - A) &= \det \begin{bmatrix} x & -1 \\ -1 & x \end{bmatrix} \det \begin{bmatrix} x-1 & 1 \\ -1 & x-1 \end{bmatrix} \\ &= (x^2 - 1)((x-1)^2 + 1) \\ &= (x-1)(x+1)(x^2 + 2x + 2). \end{aligned}$$

This has distinct prime factors so the elementary divisors of $xI - A$ are $f_1 = f_2 = f_3 = 1$ and $f_4 = (x-1)(x+1)(x^2+2x+2)$. (This is because $f_1 \mid f_2 \mid f_3 \mid f_4$ and $f_1 f_2 f_3 f_4 = (x-1)(x+1)(x^2+2x+2)$. So if $f_3 \neq 1$, then the then $f_3^2 \mid (x-1)(x+1)(x^2+2x+2)$ so that the square of some non-constant polynomial would divide $(x-1)(x+1)(x^2+2x+2)$. As this is not the case we have $f_3 = 1$.) □

(b) The elementary divisors.

Solution: The elementary divisors are the prime power factors of the invariant factors. So in this case they are $q_1 = x - 1$, $q_2 = x + 1$, $q_3 = x^2 - 2x + 2$. □

(c) The minimal polynomial.

Solution: The minimal polynomial is just the invariant factor of largest degree. That is $\min_A(x) = f_x = (x-1)(x+1)(x^2+2x+2)$. □

(d) The rational canonical form over the reals.

Solution: The rational canonical form is the block diagonal matrix with the companion matrices of the elementary divisors along the diagonal. This is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

6. Let V be a finite dimensional vector space over the real numbers $\mathbf{R}$ and let W be a subspace of V. Assume that there are vectors $v_1, v_2 \in V$ so that

$$v_1 \notin W \qquad \text{and} \qquad v_2 \notin \operatorname{Span}(\{v_1\} \cup W\}).$$

There show there is a linear functional $f \in V^*$ so that $f(w) = 0$ for all $w \in W$, $f(v_1) = 1$ and $f(v_2) = 2$.

Solution: We know that a given in finite dimensional vector spaces we can separate a vector not in a subspace form a subspace by a linear functional. This means that we can find linear functionals $f_1, f_2 \in V^*$ so that

$$f_1(v_1) = 1, \quad \text{and} \quad f_1(w) = 0 \quad \text{for all} \quad w \in W$$

and

$$f_2(v_2) = 1 \quad \text{and} \quad f_2(x) = 0 \quad \text{for all} \quad x \in \operatorname{Span}(\{v_1\} \cup W\}).$$

Let f be the linear combination of f_1 and f_2 given by

$$f = f_1 + (2 - f_1(v_2))f_2.$$

(Note that $(2 - f_1(v_2))$ is a scalar so this is a linear combination.) Then if $w \in W$ we have

$$f(w) = f_1(w) + (2 - f_1(v_2))f_2(w) = 0$$

as f_1 and f_2 are both zero on W. Next

$$f(v_1) = f_1(v_1) + (2 - f_1(v_2))f_2(v_1) = 1 + 0 = 1$$

as $f_1(v_1) = 1$ and $f_2(v_1) = 0$. And to finish

$$f(v_2) = f_1(v_1) + (2 - f_1(v_2))f_2(v_2) = f_1(v_1) + (2 - f_1(v_2)) = 2$$

as $f_2(v_2) = 1$. □

7. Let V be a finite dimensional vector space over the complex numbers and $P\colon V \to V$ a linear map so that $P^2 = P$. Show that $\operatorname{trace} P = \operatorname{rank} P$

Solution: We know that if $P^2 = P$ (that is P is a projection) that $V = \operatorname{image} P \oplus \ker P$. (This can be seen directly as for $v \in V$ we have $v = Pv + (v - Pv)$ and $P(v - Pv) = Pv - P^2 v = Pv - Pv = 0$ so that $v - Pv \in \ker P$. Therefore $\operatorname{image} P + \ker P = V$. But if $v \in \operatorname{image} P \cap \ker P$ then we have $v = Px$ for some x. Then $Pv = P^2 x = Px = v$. Therefore, as also $v \in \operatorname{image} P$, $v = Pv = 0$. Thus $\operatorname{image} P \cap \ker P = \{0\}$. But $\operatorname{image} P + \ker P = V$ and $\operatorname{image} P \cap \ker P = \{0\}$ imply that $V = V = \operatorname{image} P \oplus \ker P$. Let $n = \dim V$ and $r = \dim \operatorname{image} P = \operatorname{rank} P$ and choose a basis $v_1, \ldots, v_r$ of $\operatorname{image} P$ and then a basis of $v_{r+1}, \ldots, v_n$ of $\ker P$. Then $\{v_1, \ldots, v_n\}$ is a basis of V. If $1 \le i \le r$ then $v_i \in \operatorname{image} P$ so that $v_i = Px_i$ for some $x_i \in V$. But then $Pv_i = P^2 x_i = Px_i = v_i$. If $r + 1 \le j \le n$ then $v_j \in \ker P$ and so $Pv_j = 0$. Summarizing:

$$Pv_i = \begin{cases} v_i, & 1 \le i \le r; \\ 0, & r + 1 \le i \le n. \end{cases}$$

This implies the matrix of P in the basis $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the block matrix

$$[P]_{\mathcal{V}} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where I_r is the $r \times r$ identity matrix. But then $\operatorname{trace} P = \operatorname{trace}[P]_{\mathcal{V}} = r = \operatorname{rank} P$. □

8. Let $\mathcal{P}_2 = \operatorname{Span}\{1, x, x^2\}$ be the real polynomials of degree ≤ 2. Define $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ by

$$T(p)(x) = p(x + 1).$$

Let $\mathcal{P}_2^*$ be the dual space to $\mathcal{P}_2$ and let $\Lambda \in \mathcal{P}_2^*$ be the functional

$$\Lambda(p) = p(-3).$$

Then compute $\langle x^2, T^*\Lambda\rangle$.

Solution: This is just a case through the definitions:

$$\begin{aligned} \langle x^2, T^*\Lambda\rangle &= \langle Tx^2, \Lambda\rangle \\ &= \langle (x+1)^2, \Lambda\rangle \\ &= (-3 + 1)^2 \\ &= 4. \end{aligned}$$

9. A 5×5 real matrix A has minimal polynomial $\min_A(x) = (x-1)^2(x-2)(x-3)$ and $\operatorname{trace}(A) = 10$. What is the rational canonical form for A?

Solution: Let $f_1, \ldots, f_5$ be the invariant factors of A. Then f_5 is the minimal polynomial of A and so $f_5 = (x-1)^2(x-2)(x-3)$. We also know that the product $f_1 f_2 f_3 f_4 f_5 = \operatorname{char}_A(x) = \det(xI - A)$ which has degree 5. Therefore the only possibilities for the other invariant factors are $f_1 = f_2 = f_3 = 1$ and f_3 is one of $x-1$, $x-2$, x_3. Let $f_4 = x - a$ were a (which is either 1, 2, or 3) is to be determined. Then the elementary divisors of A will be $(x-1)^2 = x^2 - 2x + 1$, $x-2$, $x-3$ and $x-a$ so that the rational canonical form is

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

The trace of this is $0 + 2 + 2 + 3 + a = 7 + a$ and we are given that the trace is 10. Therefore $a = 3$ and the rational cononical form of A is

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

10. Let V and W be vector spaces over the field $\mathbf{F}$ and let $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$ so the following tow conditions hold
(a) $\operatorname{Span}\{v_1, \ldots, v_n\} = V$ (but we do not assume $\{v_1, \ldots, v_n\}$ is linearly independent),
(b) For any scalars $c_1, \ldots, c_n \in \mathbf{F}$

(2) $$\sum_{k=1}^{n} c_k v_k = 0 \qquad \text{implies} \qquad \sum_{k=1}^{n} c_k w_k = 0.$$

Show that there is a unique linear map $T\colon V \to W$ so that $Tv_k = w_k$ for $1 \le k \le n$.

Solution: By reordering we can assume that $v_1, \ldots, v_k$ is a basis of V where $k \le n$. Then for $k+1 \le j \le n$ we have that $v_j \in \operatorname{Span}\{v_1, \ldots, v_k\}$. By our basic existence theorem for linear maps we that that there is a unique linear map $T\colon V \to W$ so that

$$Tv_i = w_i \quad \text{for} \quad 1 \le i \le k.$$

(Note that if the required map exists that it must satisfy this condition. This shows that if T exists, then it is unique.) For $k+1 \le j \le n$ we have that $v_j \in \operatorname{Span}\{v_1, \ldots, v_k\}$ there are scalars $c_1, \ldots, c_k$ so that $v_j = c_1 v_1 + \cdots c_k v_k$. Therefore

(3) $$Tv_j = T(c_1 v_1 + \cdots c_k v_k) = c_1 T v_1 + \cdots + c_k T v_k = c_1 w_1 + \cdots + c_k w_k.$$

But we also have

$$v_j - c_1 v_1 - \cdots - c_k v_k = 0.$$

By the condition (2) this implies $w_j - c_1 w_1 - \cdots - c_k w_k = 0$ so that

(4) $$w_j = c_1 w_1 + \cdots + c_k w_k.$$

Now combining (3) and (4) gives that $Tv_j = w_j$. This shows that $Tv_i = w_i$ for $1 \leq i \leq n$ and completes the proof.