Mathematics 700 Quiz #2 Name: **Answer Key**

In what follows let $\mathcal{P}_2$ be the real polynomials of degree ≤ 2. We work with the two bases

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{B}' = \{1, (x-1), (x-1)^2\}.$$

Let $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ be

$$(Tp)(x) = x^2 p\left(\frac{x-1}{x}\right).$$

Remark: Some of you interpreted the definition of T to mean

$$(Tp)(x) = x^2 \times p(x) \times \left(\frac{x-1}{x}\right).$$

This is not consistent with standard mathematical usage and is certainly not consistent with past examples we have done in class and on homework.

1. Find the change of basis matrix P between the bases $\mathcal{B}$ and $\mathcal{B}'$.

Solution: For this and the following problems we will use the notation:

$$p_0(x) = 1,\ p_1(x) = x,\ p_2(x) = x^2, \qquad p_0'(x) = 1,\ p_1'(x) = (x-1),\ p_2'(x) = (x-1)^2.$$

so that

$$\mathcal{B} = \{p_0, p_1, p_2\} \quad \text{and} \quad \mathcal{B}' = \{p_0', p_1', p_2'\}.$$

The change of basis matrix P between two $\mathcal{B}$ and $\mathcal{B}'$ is the matrix whose columns are the coordinate vectors of $\mathcal{B}'$ with respect to the basis $\mathcal{B}'$. In the case at hand let $P = [P_1, P_2, P_3]$. So express p_0', p_1', and p_2' with to $\mathcal{B}$.

$$p_0'(x) = 1 = (Tp_0')(x) = 0p_0 + 0p_1 + 1p_2, \quad \text{so} \quad P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$p_1'(x) = x - 1 = -1p_0 + 1p_1 + 0p_2, \quad \text{so} \quad P_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

$$p_2'(x) = (x-1) = x^2 - 2x + 1 = 1p_0 - 2p_1 + 1p_2, \quad \text{so} \quad P_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Thus

$$P = [P_1, P_2, P_3] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. Find the matrix $A := [T]_{\mathcal{B}}$ for T in the basis $\mathcal{B}$.

Solution: The matrix of a linear map with respect to a basis has as columns the image of the basis under the linear map expressed in terms of the basis. Letting $A = [A_1, A_2, A_3]$ with A_1, A_2 and A_3 the columns of A we then have

$$(Tp_0)(x) = x^2 p_0\left(\frac{x-1}{x}\right) = x^2 1 = 0p_0 + 0p_1 + 1p_2$$

so that

$$A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

Acting on the second vector of $\mathcal{B}$ gives

$$(Tp_1)(x) = x^2 p_1\left(\frac{x-1}{x}\right) = x^2\left(\frac{x-1}{x}\right) = x^2 - x = 0p_0 - 1p_1 + 1p_2$$

so that

$$A_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

and acting on the third vector of $\mathcal{B}$ gives

$$(Tp_2)(x) = x^2 p_2\left(\frac{x-1}{x}\right) = x^2\left(\frac{x-1}{x}\right)^2 = x^2 - 2x + 1 = 1p_0 - 2p_1 + 1p_2$$

so that

$$A_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Therefore

$$A = [A_1, A_2, A_3] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

3. Find the matrix $B := [T]_{\mathcal{B}'}$ for T in the basis $\mathcal{B}'$.

Solution: This works like the last problem, but remenber that in compution the coordinates in terms of the basis $\mathcal{B}'$ that Tp_i' is written as a linear combinatin of the elments of $\mathcal{B}'$, that is 1, $(x-1)$ and $(x-1)^2$, and *not* in terms of 1, x and x^2. Letting $B = [B_1, B_2, B_3]$ so that B_1, B_2, and B_3 are the columns of B we have

$$(Tp_0')(x) = x^2 p_0'\left(\frac{x-1}{x}\right) = x^2 1 = (x-1+1)^2 = (x-1)^2 + 2(x-1) + 1 = 1p_0' + 2p_1' + 1p_2'$$

so that

$$B_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

Acting on the second vector of $\mathcal{B}'$ gives

$$(Tp_1')(x) = x^2 p_1'\left(\frac{x-1}{x}\right) = x^2\left(\frac{x-1}{x} - 1\right) = x(x-1) - x^2 = -(x-1) - 1 = -1p_0' - 1p_1' + 0p_2'$$

so that

$$B_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

and acting on the third vector of $\mathcal{B}'$ gives

$$(Tp_2')(x) = x^2 p_2'\left(\frac{x-1}{x}\right) = x^2\left(\frac{x-1}{x} - 1\right)^2 = (x-1-x)^2 = 1 = 1p_0 + 0p_1 + 0p_2$$

so that

$$B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$B = [B_1, B_2, B_3] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

4. Check your work in the first three parts of this by showing $B = P^{-1}AP$.

SOLUTION: **Labor saving observation:** The equation $B = P^{-1}AP$ is equivalent to $PB = AP$ so that you do not have to compute the inverse of P. Computing these products:

$$AP = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and

$$PB = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Therefore it works out.

If you did comute P^{-1} and want a check here it is

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$