Mathematics 700 Homework  
Due Monday, October 25

**Problem 1.** This problem is to familiarize you with some properties of multiplication by diagonal matrices with are more or less obvious after seeing them. Let

\[ D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}. \]

(a) Let \( A \) be a matrix with \( n \) rows (and any number of columns)

\[ A = \begin{bmatrix}
A^1 \\
A^2 \\
\vdots \\
A^n
\end{bmatrix}. \]

Then mutiplying \( A \) on the left by \( D \) multiplies the rows of by \( \lambda_1, \lambda_2, \ldots, \lambda_n \). That is you are to show

\[ DA = \begin{bmatrix}
\lambda_1 A^1 \\
\lambda_2 A^2 \\
\vdots \\
\lambda_n A^n
\end{bmatrix}. \]

(b) Likewise show that if \( B \) has \( n \) columns (and any number of rows)

\[ B = \begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_n
\end{bmatrix} \]

then multiplying \( B \) on the right by \( D \) multiplies the columns by \( \lambda_1, \ldots, \lambda_n \). That is

\[ BD = \begin{bmatrix}
\lambda_1 B_1 \\
\lambda_2 B_2 \\
\vdots \\
\lambda_n B_n
\end{bmatrix}. \]

(c) Assume that \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are all distinct. Then find all \( n \times n \) matrices \( C \) which commute with \( D \). That is all \( C \) so that \( CD = DC \). □

Polynomials are going to play a larger and larger rôle in our understanding of matrices and linear operators. Let \( \mathbf{F} \) be our field of scalars. Then we use the standard notation \( \mathbf{F}[x] \) for the set of all polynomials in the indeterminate \( x \) over \( \mathbf{F} \). That is \( p(x) \in \mathbf{F}[x] \) is an expression of the form

\[ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \]

We add and multiply these by the rules you have been using for as long as you can remember. If \( A \) is a square matrix and \( p(x) \in \mathbf{F}[x] \) is given by (1) then we can define \( p(A) \) to be \( p(x) \) evaluated at \( x = A \). That is

\[ p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n \]
where \( I \) is the identity matrix. By convention \( A^0 = I \) for all matrices. Likewise if \( T: V \to V \) is a linear operator on a vector space \( V \) then we define
\[
p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n
\]
where this time \( I \) is the identity map on \( V \) and again \( T^0 = I \) by convention.
As an example let
\[
A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}
\]
and \( p(x) = 2x^2 - 4x + 3 \). Then
\[
p(A) = 2A^2 - 4A + 3I
\]
\[
= 2 \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}^2 - 4 \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
= 2 \begin{bmatrix} -5 & -12 \\ 18 & 19 \end{bmatrix} - 4 \begin{bmatrix} -4 & 8 \\ -12 & -20 \end{bmatrix} + 3 \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}
\]
\[
= \begin{bmatrix} -10 & -24 \\ 36 & 38 \end{bmatrix} + \begin{bmatrix} -1 & 8 \\ -12 & -17 \end{bmatrix}
\]
\[
= \begin{bmatrix} -11 & -16 \\ 24 & 21 \end{bmatrix}.
\]

**Problem 2.** Let \( D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \).
(a) Show \( p(D) = \text{diag}(p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n)) \).
(b) If \( f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n) \), then what is \( f(D) \)? \( \square \)

**Problem 3.** Let \( A, B \in M_{n \times n}(F) \) and assume that \( A \) and \( B \) commute (that is \( AB = BA \)). Let \( p(x) \in F[x] \) be a polynomial. Show that \( p(A) \) and \( B \) commute. \( \square \)

**Problem 4.** Let \( A \in M_{n \times n}(F) \) be a square matrix. Then show that there is a nonzero polynomial \( p(x) \in F[x] \) of degree \( \leq n^2 + 1 \) so that \( p(A) = 0 \). (We will latter do much better and show \( p(x) \) can be chosen with \( \text{deg } p(x) \leq n \). A first step in this direction is Problem 6 below.) *HINT:* Consider \( \{ I, A, A^2, \ldots, A^{n^2}, A^{n^2+1} \} \) and use that \( \dim M_{n \times n}(F) = n^2 \). \( \square \)

**Problem 5.** Let \( A, B \in M_{n \times n}(F) \) be similar. Then show that for any polynomial \( p(x) \in F[x] \) that \( p(A) \) and \( p(B) \) are similar. Specifically show that if \( B = P^{-1}AP \) then \( p(B) = P^{-1}p(A)P \). \( \square \)

**Problem 6.** Let \( A \in M_{n \times n}(F) \) be a square matrix and assume that the roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of the characteristic polynomial \( \text{char}_A(x) = \det(xI - A) \) are distinct. Let \( f(x) \) be the polynomial
\[
f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n).
\]
(A little thought will should convince you that \( f(x) = \text{char}_A(x) \), but this is not needed to do the problem.) Then show \( f(A) = 0 \). *HINT:* Recall that if the roots of \( \text{char}_A(x) \) are distinct, then \( A \) is diagonalizable. Now use Problems 2 (b) and 5. \( \square \)
**Problem 7.** Let $T: V \to V$ be a linear map and assume $\lambda$ is an eigenvalue of $T$. This show that for any polynomial $p(x) \in \mathbb{F}[x]$ that $p(\lambda)$ is an eigenvalue of $p(T)$.

**Hint:** Consider the special case of $p(x) = x^2 + 3$. Let $v \in V$ be an eigenvector for $\lambda$, that is $v \neq 0$ and $Tv = \lambda v$. Then $p(T)v = (T^2 + 3I)v = TTv + 3Iv = T\lambda v + 3v = \lambda Tv + 3v = \lambda^2 v + 3v = (\lambda^2 + 3)v = p(\lambda)v$. $\square$

Here are some problems related to linear functionals and hyperplanes. (The relation between the two being that hyperplanes are the kernels of nonzero linear functionals and that the kernel of a linear functional determines it up to multiplication by a scalar.)

**Problem 8.** Let $W$ be a hyperplane in the finite dimensional vector space $V$ and let $v_0 \in V$ with $v_0 \notin W$. Then show that every vector $v \in V$ can be uniquely written as $v = w + cv_0$ where $w \in W$ and $c \in \mathbb{F}$ is a scalar. $\square$

**Problem 9.** Let $M_{2\times2}(\mathbb{R})$ be the vector space of $2 \times 2$ matrices over the real numbers $\mathbb{R}$. Let $f: M_{2\times2}(\mathbb{R}) \to \mathbb{R}$ be a linear functional so that $f(AB) = f(BA)$ for all $A, B \in M_{2\times2}(\mathbb{R})$. Show there is a scalar $c \in \mathbb{R}$ so that $f(A) = c \operatorname{tr}(A)$ for all $A \in M_{2\times2}(\mathbb{R})$.

**Hint:** Both $f$ and $\operatorname{tr}$ are linear functionals on $M_{2\times2}(\mathbb{R})$ so it is enough to show that $\ker(\operatorname{tr}) \subseteq \ker(f)$. Show that $\ker(\operatorname{tr}) = \operatorname{Span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. Now $f(AB) = f(BA)$ and linearity imply that $f(AB - BA) = 0$ for all $A, B \in M_{2\times2}(\mathbb{R})$. Use this to show that $f(C) = 0$ for $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. (You should compare this with Problem 2 on the January 1985 qualifying exam.)

Finally here is a problem to review on quotient spaces.

**Problem 10.** Let $V$ be a vector space and $W \subset V$ a subspace. Let $T: V \to V$ be linear and assume that $W$ is invariant under $T$. (This means that $T[W] \subseteq W$ or what is the same thing if $v \in W$ the $Tv \in W$. For any $v \in V$ let $[v]$ be the coset of $W$ in $V$. (That is $[v] = v + W = \{x \in V : v - x \in W\}$. As before $[x] = [y]$ if and only if $x - y \in W$.) Define $\widehat{T}: V/W \to V/W$ by $\widehat{T}[x] = [Tx]$.

(a) Show that $\widehat{T}$ is well defined. That is show that if $[x] = [y]$ then $[Tx] = [Ty]$.

(b) Show $T$ is linear.