Mathematics 700 Homework
Due Monday, October 25

Problem 1. This problem is to familiarize you with some properties of multiplication by diagonal matrices with are more or less obvious after seeing them. Let

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

(a) Let A be a matrix with n rows (and any number of columns)

$$A = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^n \end{bmatrix}.$$

Then mutiplying A on the left by D multiplies the rows of by $\lambda_1, \lambda_2, \dots, \lambda_n$. That is you are to show

$$DA = \begin{bmatrix} \lambda_1 A^1 \\ \lambda_2 A^2 \\ \vdots \\ \lambda_n A^n \end{bmatrix}.$$

(b) Likewise show that if B has n columns (and any number of rows)

$$B = \begin{bmatrix} B_1, B_2, \dots, B_n \end{bmatrix}$$

then multiplying B on the right by D multiplies the columns by $\lambda_1, \dots, \lambda_n$. That is

$$BD = \begin{bmatrix} \lambda_1 B_1, \lambda_2 B_2, \dots, \lambda_n B_n \end{bmatrix}.$$

(c) Assume that $\lambda_1, \lambda_2, \dots, \lambda_n$ are all distinct. Then find all $n \times n$ matrices C which commute with D. That is all C so that $CD = DC$. □

Polynomials are going to play a larger and larger rôle in our understanding of matrices and linear operators. Let $\mathbf{F}$ be our field of scalars. Then we use the standard notation $\mathbf{F}[x]$ for the set of all polynomials in the indeterminate x over $\mathbf{F}$. That is $p(x) \in \mathbf{F}[x]$ is an expression of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \tag{1}$$

We add and multiply these by the rules you have been using for as long as you can remember. If A is a square matrix and $p(x) \in \mathbf{F}[x]$ is given by (1) then we can define $p(A)$ to be $p(x)$ evaluated at $x = A$. That is

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$$

where I is the identity matrix. By convention $A^0 = I$ for all matrices. Likewise if $T\colon V \to V$ is a linear operator on a vector space V then we define

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n$$

where this time I is the identity map on V and again $T^0 = I$ by convention.

As an example let

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$$

and $p(x) = 2x^2 - 4x + 3$. Then

$$\begin{aligned} p(A) &= 2A^2 - 4A + 3I \\ &= 2\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}^2 - 4\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 2\begin{bmatrix} -5 & -12 \\ 18 & 19 \end{bmatrix} + \begin{bmatrix} -4 & 8 \\ -12 & -20 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -10 & -24 \\ 36 & 38 \end{bmatrix} + \begin{bmatrix} -1 & 8 \\ -12 & -17 \end{bmatrix} \\ &= \begin{bmatrix} -11 & -16 \\ 24 & 21 \end{bmatrix}. \end{aligned}$$

Problem 2. Let $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

(a) Show $p(D) = \operatorname{diag}(p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n))$.

(b) If $f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$, then what is $f(D)$? □

Problem 3. Let $A, B \in M_{n\times n}(\mathbf{F})$ and assume that A and B commute (that is $AB = BA$). Let $p(x) \in \mathbf{F}[x]$ be a polynomial. Show that $p(A)$ and B commute. □

Problem 4. Let $A \in M_{n\times n}(\mathbf{F})$ be a square matrix. Then show that there is a nonzero polynomial $p(x) \in \mathbf{F}[x]$ of degree $\leq n^2 + 1$ so that $p(A) = 0$. (We will latter do much better and show $p(x)$ can be chosen with $\deg p(x) \leq n$. A first step in this direction is Problem 6 below.) Hint: Consider $\{I, A, A^2, \ldots, A^{n^2}, A^{n^2+1}\}$ and use that $\dim M_{n\times n}(\mathbf{F}) = n^2$. □

Problem 5. Let $A, B \in M_{n\times n}(\mathbf{F})$ be similar. Then show that for any polynomial $p(x) \in \mathbf{F}[x]$ that $p(A)$ and $p(B)$ are similar. Specifically show that if $B = P^{-1}AP$ then $p(B) = P^{-1}p(A)P$. □

Problem 6. Let $A \in M_{n\times n}(\mathbf{F})$ be a square matrix and assume that the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic polynomial $\operatorname{char}_A(x) = \det(xI - A)$ are distinct. Let $f(x)$ be the polynomial

$$f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n).$$

(A little thought will should convince you that $f(x) = \operatorname{char}_A(x)$, but this is not needed to do the problem.) Then show $f(A) = 0$. Hint: Recall that if the roots of $\operatorname{char}_A(x)$ are distinct, then A is diagonalizable. Now use Problems 2 (b) and 5. □

Problem 7. Let $T\colon V \to V$ be a linear map and assume λ is an eigenvalue of T. This show that for any polynomial $p(x) \in \mathbf{F}[x]$ that $p(\lambda)$ is an eigenvalue of $p(T)$. HINT: Consider the special case of $p(x) = x^2 + 3$. Let $v \in V$ be an eigenvector for λ, that is $v \neq 0$ and $Tv = \lambda v$. Then $p(T)v = (T^2 + 3I)v = TTv + 3Iv = T\lambda v + 3v = \lambda Tv + 3v = \lambda\lambda v + 3v = (\lambda^2 + 3)v = p(\lambda)v$. $\square$

Here are some problems related to linear functionals and hyperplanes. (The relation between the two being that hyperplanes are the kernels of nonzero linear functionals and that the kernel of a linear functional determines it up to multiplication by a scalar.)

Problem 8. Let W be a hyperplane in the finite dimensional vector space V and let $v_0 \in V$ with $v_0 \notin W$. Then show that every vector $v \in V$ can be uniquely written as $v = w + cv_0$ where $w \in W$ and $c \in \mathbf{F}$ is a scalar. $\square$

Problem 9. Let $M_{2\times 2}(\mathbf{R})$ be the vector space of 2×2 matrices over the real numbers $\mathbf{R}$. Let $f\colon M_{2\times 2}(\mathbf{R}) \to \mathbf{R}$ be a linear functional so that $f(AB) = f(BA)$ for all $A, B \in M_{2\times 2}(\mathbf{R})$. Show there is a scalar $c \in \mathbf{R}$ so that $f(A) = c\,\mathrm{tr}(A)$ for all $A \in M_{2\times 2}(\mathbf{R})$. HINT: Both f and tr are linear functionals on $M_{2\times 2}(\mathbf{R})$ so it is enough to show that $\ker(\mathrm{tr}) \subseteq \ker(f)$. Show that $\ker(\mathrm{tr}) = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. Now $f(AB) = f(BA)$ and linearity imply that $f(AB - BA) = 0$ for all $A, B \in M_{2\times 2}(\mathbf{R})$. Use this to show that $f(C) = 0$ for $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. (You should compare this with Problem 2 on the January 1985 qualifying exam.)

Finally here is a problem to review on quotient spaces.

Problem 10. Let V be a vector space and $W \subset V$ a subspace. Let $T\colon V \to V$ be linear and assume that W is invariant under T. (This means that $T[W] \subseteq W$ or what is the same thing if $v \in W$ the $Tv \in W$. For any $v \in V$ let $[v]$ be the coset of W in V. (That is $[v] = v + W = \{x \in V : v - x \in W\}$. As before $[x] = [y]$ if and only if $x - y \in W$.) Define $\widehat{T}\colon V/W \to V/W$ by

$$\widehat{T}[x] = [Tx].$$

(a) Show that $\widehat{T}$ is well defined. That is show that if $[x] = [y]$ then $[Tx] = [Ty]$.
(b) Show $\widehat{T}$ is linear.