Mathematics 700 Homework  
Due Monday, October 18

**Problem 1.** Find the trace and determinant of the following linear maps or matrices.

(a) The matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 3 \\ 7 & 5 & 4 \end{bmatrix}$

(b) The linear map $T: \mathcal{P}_2 \to \mathcal{P}_2$ given by

$$T_p := \frac{d^2}{dx^2}((x^2 + 1)p(x)).$$

(c) The linear map $L: M_{2\times 2} \to M_{2\times 2}$ given by $L(X) = AX$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}$. HINT: I found the easiest basis for $M_{2\times 2}$ to use in this problem was $B = \{E_{11}, E_{21}, E_{12}, E_{22}\}$ where $E_{ij}$ has a one in the $i-j$ entry and zeros elsewhere.

**Problem 2.** Let $T: V \to V$ be a linear map and let $\lambda$ be an eigenvalue of $T$. Then the eigenspace of $\lambda$ is $V_\lambda := \{v \in V : Tv = \lambda v\}$. Show that $V_\lambda = \ker(T - \lambda I)$ and conclude that $V_\lambda$ is a subspace of $V$. (You can use that $\ker(T - \lambda I)$ is a subspace.)

**Problem 3.** Two linear maps $S, T: V \to V$ **commute** if $ST = TS$. Show that if $S$ and $T$ commute and if $\lambda$ is an eigenvalue of $T$ then the eigenspace $V_\lambda = \ker(T - \lambda I)$ is invariant under $S$. (That is if $v \in V_\lambda$ then $Sv \in V_\lambda$.)

**Problem 4.** (a) Problem 4 (a) from the August 1989 qualifying exam.

(b) Problem 4 (a) from the August 1990 qualifying exam.

**Problem 5.** Show that the characteristic polynomial of a $2\times 2$ matrix $A$ is $\chi_A(x) = x^2 - \text{tr}(A)x + \det(A)$.

**Problem 6.** Show that if $A$ is a $2 \times 2$ matrix then $A^2 - \text{tr}(A)A + \det(A)I = 0$. (In light of the last problem this implies that $A$ is a root of its characteristic polynomial.)

**Problem 7.** Find eigenvalues and eigenvectors of the following linear maps or matrices.

(a) The matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(b) The matrix $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.

(c) The linear map $S: \mathcal{P}_2 \to \mathcal{P}_2$ given by $(Sp)(x) = (x + 1)^2p \left(\frac{x-1}{x+1}\right)$.

**Problem 8.** Let $A = [a_{ij}]$ be an $n \times n$ matrix over the complex numbers. Then it is nice to be able to estimate the eigenvalues of $A$ without having to compute them exactly. Here is a standard result about the location of the eigenvalues. For each
Let $i \in \{1, \ldots, n\}$ let $r_i$ be the sum of the absolute values of the off diagonal elements in the $i$-th row. That is

$$r_i := \sum_{j\neq i} |a_{ij}|.$$ 

Let $D_i$ be the closed disk of radius $r_i$ centered at $a_{ii}$. Explicitly:

$$D_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\}.$$ 

Prove if $\lambda$ is an eigenvalue of $A$ then $\lambda \in \bigcup_{i=1}^{n} D_i$. HINT: If $\lambda$ is an eigenvalue of $A$ then there is a nonzero column vector $v = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ so that $Av = \lambda v$. That is

$$\sum_{j=1}^{n} a_{ij} z_j = \lambda z_i \quad \text{for} \quad i = 1, \ldots, n.$$ 

Choose $i_0$ to maximize $|z_j|$. That is $|z_j| \leq |z_{i_0}|$ for $j = 1, \ldots, n$. Then let $i = i_0$ in (1) and rewrite as

$$(\lambda - a_{i_0 i_0}) z_{i_0} = \sum_{j \neq i_0} a_{ij} z_j.$$ 

Now take the absolute value of both sides of this equation and use the triangle inequality and that $|z_j| \leq |z_{i_0}|$ to conclude $|\lambda - a_{i_0 i_0}| \leq r_{i_0}.$

**Problem 9.** Problem 5 on the January 1993 qualifying exam. HINT: If $\det A = 0$ then $A$ has $\lambda = 0$ as an eigenvalue. Use the last problem to show this is impossible.