Mathematics 700 Homework
Due Monday, October 18

Problem 1. Find the trace and determinant of the following linear maps or matrices.

(a) The matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 3 \\ 7 & 5 & 4 \end{bmatrix}$

(b) The linear map $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ given by

$$Tp := \frac{d^2}{dx^2}((x^2+1)p(x)).$$

(c) The linear map $L\colon M_{2\times 2} \to M_{2\times 2}$ given by $L(X) = AX$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}$. HINT: I found the easiest basis for $M_{2\times 2}$ to use in this problem was $\mathcal{B} = \{E_{11}, E_{21}, E_{12}, E_{22}\}$ where E_{ij} has a one in the i-j entry and zeros elsewhere.

Problem 2. Let $T\colon V \to V$ be a linear map and let λ be an eigenvalue of T. Then the eigenspace of of λ is $V_\lambda := \{v \in V : Tv = \lambda v\}$. Show that $V_\lambda = \ker(T - \lambda I)$ and conclude that V_λ is a subspace of V. (You can use that $\ker(T - \lambda I)$ is a subspace.)

Problem 3. Two linear maps $S, T\colon V \to V$ ***commute*** iff $ST = TS$. Show that if S and T commute and if λ is an eigenvalue of T then the eigenspace $V_\lambda = \ker(T - \lambda I)$ is invariant under S. (That is if $v \in V_\lambda$ then $Sv \in V_\lambda$.)

Problem 4. (a) Problem 4 (a) from the August 1989 qualifying exam.
(b) Problem 4 (a) from the August 1990 qualifying exam.

Problem 5. Show that the characteristic polyniomial of a 2×2 matrix A is $\operatorname{char}_A(x) = x^2 - \operatorname{tr}(A)x + \det(A)$.

Problem 6. Show that if A is a 2×2 matrix then $A^2 - \operatorname{tr}(A)A + \det(A)I = 0$. (In light of the last problem this implies that A is a root of its characteristic polynomial.)

Problem 7. Find eigenvalues and eignenvectors of the following linear maps or matrics.

(a) The matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(b) The matrix $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.

(c) The linear map $S\colon \mathcal{P}_2 \to \mathcal{P}_2$ given by $(Sp)(x) = (x+1)^2 p\left(\dfrac{x-1}{x+1}\right)$.

Problem 8. Let $A = [a_{ij}]$ be an $n \times n$ matrix over the complex numbers. Then it is nice to be able to estimate the eigenvalues of A without having to compute them exactly. Here is a standard result about the location of the eigenvalues. For each

$i \in \{1, \ldots, n\}$ let r_i be the sum of the absolute values of the off diagonal elements in the i-th row. That is

$$r_i := \sum_{j \neq i} |a_{ij}|.$$

Let D_i be the closed disk of radius r_i centered at a_{ii}. Explicitly:

$$D_i = \{z \in \mathbf{C} : |z - a_{ii}| \leq r_i\}.$$

Prove if λ is an eigenvalue of A then $\lambda \in \bigcup_{i=1}^n D_i$. HINT: If λ is an eigenvalue of A then then is a nonzero column vector $v = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ so that $Av = \lambda v$. That is

$$\sum_{j=1}^{n} a_{ij} z_j = \lambda z_i \quad \text{for} \quad i = 1, \ldots, n. \tag{1}$$

Choose i_0 to maximize $|z_j|$. That is $|z_j| \leq |z_{i_0}|$ for $j = 1, \ldots, n$. Then let $i = i_0$ in (1) and rewrite as

$$(\lambda - a_{i_0 i_0}) z_{i_0} = \sum_{j \neq i_0} a_{ij} z_j.$$

Now take the absolute value of both sides of this equation and use the triangle inequality and that $|z_j| \leq |z_{i_0}|$ to conclude $|\lambda - a_{i_0 i_0}| \leq r_{i_0}$.

Problem 9. Problem 5 on the January 1993 qualifying exam. HINT: If $\det A = 0$ then A has $\lambda = 0$ as an eigenvalue. Use the last problem to show this is impossable.