Mathematics 700 Homework
Due Wednesday, October 13

Some more on matrices of linear maps. The following shows that it is possible to choose a bases for the range and domain of a linear map that puts makes the matrix particularly simple.

Problem 1. Let $V$ and $W$ be finite dimensional vector spaces and in let $T: V \rightarrow W$ be a linear.

1. First assume that $\dim V = 5$, $\dim W = 6$ and that $\text{rank } T = 3$. Then by rank plus nullity we have $\text{nullity } T = 2$. Choose a basis $v_4, v_5$ of $\ker T$. Then these can be extended to a basis $V = \{v_1, v_2, v_3, v_4, v_5\}$ of $V$ (so the last two in the list are the basis of $\ker T$.) Let $w_1 = Tv_1, w_2 = Tv_2, w_3 = Tv_3$. Than as $\text{Span}\{v_1, v_2, v_3\} \cap \ker T = \{0\}$ it follows that $w_1, w_2, w_3$ are linearly independent. Thus we can extend $\{w_1, w_2, w_3\}$ to a basis $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ of $W$. Then show that in these bases the matrix of $T$ is

$$
[T]_V^W = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
$$

2. Now assume that $\dim V = m$, $\dim W = n$, and that $\text{rank } T = r$. Then show show that it is possible to choose bases $V$ for $V$ and $W$ for $W$ so that the matrix $[T]_V^W$ has $r$ ones down the main diagonal and all other element zero.

Quotient Spaces. If there is any one idea the characterizes modern algebra it is the idea of a quotient structure. The following problems introduce the linear algebra version of this concept. You may have seen other versions as in the integers mod $n$, the quotient of a group by a normal subgroup, and the quotient of a ring by an ideal.

Let $V$ be a vector space over the field $\mathbb{F}$ and $W$ a subspace of $V$. Then define an equivalence relation $\sim_W$ by

$$v_1 \sim_W v_2 \text{ if and only if } v_2 - v_1 \in W.$$ 

Problem 1. Show that this is an equivalence relation. (Recall a relation $\sim$ on a set $V$ is an equivalence relation iff the three conditions (1) $x \sim x$ for all $x \in V$ (it is reflective) (2) $x \sim y$ implies $y \sim x$ for all $x, y \in V$ ($\sim$ is symmetric) and (3) $x \sim y$ and $y \sim z$ implies $x \sim y$ ($\sim$ is transitive) hold.)

Denote by $[v]_W$ the equivalence class of $v \in V$ under the equivalence relation $\sim_W$. That is

$$[v]_W := \{u \in V : u \sim_W v\}.$$
Problem 2. Show \([v]_W = v + W\) where \(v + W = \{v + w : w \in W\}\).

Let \(V/W\) be the set of all equivalence classes of \(\sim_W\). That is
\[
V/W := \{[v]_W : v \in V\} = \{v + W : v \in V\}.
\]
The equivalence class \([v]_W = v + W\) is often called the **coset of** \(v\) **in** \(V/W\).

Problem 3. Let \(V = \mathbb{R}^2\) and let \(W\) be the subspace of points of \(V\) of points \((x, y)\) with \(y = -2x\). Then draw pictures of the coset of \((1, 1)\) in \(V/W\) and the coset of \((3, -2)\) in \(V/W\). What is a geometric description of the coset of \(v \in \mathbb{R}^2\) in \(V/W\)?

Define a sum and scalar multiplication in \(V/W\) by
\[
[v_1]_W + [v_2]_W := [v_1 + v_2]_W, \quad c[v]_W := [cv]_W
\]
where \(v_1, v_2, v \in V\) and \(c \in \mathbb{F}\).

Problem 4. Show this is well defined. Recall the term **well defined** is used in mathematics to mean “is independent of the choices made in the definition”. In this particular case this means you need to show

\[
[v_1]_W = [v'_1]_W \quad \text{and} \quad [v_2]_W = [v'_2]_W \quad \text{implies} \quad [v_1 + v_2]_W = [v'_1 + v'_2]_W
\]

and
\[
[v]_W = [v']_W \quad \text{implies} \quad [cv]_W = [cv']_W.
\]

**Proposition 1.** With these operations \(V/W\) is a vector space.

Problem 5. Prove this.

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Problem 6. The natural projection \(\pi: V \to V/W\) is a linear map and \(\ker(\pi) = W\).

Problem 7. If \(V\) is finite dimensional then what is the dimension of \(V/W\) in terms of \(\dim V\) and \(\dim W\)? Prove your answer is correct. (Hint: Rank plus nullity.)

Problem 8. In the example of Problem 3 draw some pictures of cosets \(v_1 + W\) and \(v_2 + W\) what their sum \((v_1 + W) + (v_2 + W)\) and the linear combination \(2(v_1 + W) - 3(v_2 + W)\) for a few choices of \(v_1\) and \(v_2\).

We now give a very basic result. In the context of groups this is called **the first homomorphism theorem**.

**Theorem 1.** Let \(V\) and \(U\) be vector spaces and let \(T: V \to U\) be a surjective (that is onto) linear map. Then the vector space \(V/\ker T\) is isomorphic to \(U\). (Written as \(U \cong V/\ker T\).)
We are not assuming that $V$ and $U$ are finite dimensional so it is not enough to just compute dimensions.

**Problem 9.** Prove this along the following lines: First to simplify notation set $W := \ker T$ so that

$$W = \{v \in V : Tv = 0\}.$$  

Then we wish to show that $V/W \approx U$. Define a map $\tilde{T} : V/W \to U$ by

$$\tilde{T}[v]_W = Tv.$$  

(a) Show that $\tilde{T}$ is well defined. (That is if $[v_1]_W = [v_2]_W$ then $Tv_1 = Tv_2$.)

(b) Show that $\tilde{T}$ is linear.

(c) Show that $\tilde{T}$ is onto (i.e. surjective). (HINT: Use that $T$ is onto so that every $u \in U$ is of the form $u = Tv$ for some $v \in V$.)

(d) Show that $\tilde{T}$ is one to one (i.e. injective). (HINT: You wish to show that if $\tilde{T}[v]_W = 0$ then $[v]_W = [0]_W$. But $[v]_W = [0]_W$ if and only if $v \in W = \ker T$.)